GEOMETRIC AND ANALYTIC STRUCTURE OF THE YANG-MILLS THEORY

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Abstract

The birth of the Yang-Mills theory has provided such a profound intersection of complex geometric analysis and mathematical physics. The core of its theory lies on a study of connections on complex vector bundles where curvatures naturally rise to show the Yang-Mills functional; the progression toward critical points satisfies the Yang-Mills equation in turn. In this paper, we focus on complex geoemtry point of view in order to observe the foundations of Yang-Mills theory and rigorously derive the complexity of its functional, variations, and the equation.

Contents

Chapter 1: Introduction

1.1 Preface

In the early 20th century, Hermann Weyl first introduced the *gauge theory*, conjecturing that the gauge transformation(i.e.local rescaling via length) in the law of physics should be invariant; Weyl attempted to show the symmetry, in which to unify the electromagnetism with general relativity (cf.Weyl [13]). Decades later, in 1954, physicists Chen-Ning Yang and Robert Laurence Mills published a seminar article on the Physical Review, titled *Conversation of Isotopic Spin and Isotopic Gauge Invariance* [14]. The main goal in physics was to discuss the new idea of generalization of the electromagnetism to a non-Abelian gauge group. In particular, considering a simplest case, Yang and Mills proposed a theory of describing the interactions of subnuclear forces that are gauge invariant if the $U(1)$ electromagnetism replaced by $SU(2)$. This is what we call the *non-Abelian gauge theory*, and in modern terms, it is well-known as the *Yang-Mills theory*. Ever since the theory has been introduced, leading toward the evolution of theoretical physics and mathematics, including particle physics, quantum field theories, and differential and algebraic geometry. Especially, along with substantial physical development, 1970s and 1980s were prime times for the development of Yang-Mills theory from geometric perspectives led by Atiyah, Bott, Hitchin, Kobayashi, Uhlenbeck, Donaldson, Yau, and their collaborators.

This thesis, however, neither has the aim to claim significant original results nor to express new theories from previous influential contributions; but rather, focusing on the complex geoemtry point of view, it is an exposition that wishes to present a self-contained and lucid introduction to the *Yang-Mills theory*. Briefly showing the big picture, the new gauge field introduced by Yang and Mills is the *Yang-Mills field*, and through the complex geoemtry perspective, we first analyze connections and curvatures on complex vector bundles in order to gain insights. Here, the associated functional defined on connections over a complex vector bundle valued in the space of endomorphisms is called the *Yang-Mills functional*. Then, the Euler-Lagrange equation(i.e.critical points) of this functional, corresponding to connections whose curvatures lead directly toward the non-linear partial differential equation, is the *Yang-Mills equation*. Finally, the solutions of the *Yang-Mills equation* is called the *Yang-Mills field*, which is equivalent to say the connections that satisfies the equation.

1.2 Construction

The original electromagnetism and the *Yang-Mills theory* were formulated based on spacetime manifolds equipped with Lorentz metric, the Minkowski space. However, since our viewpoint is on complex geometry, we work on base arbitrary complex manifold associated with a Hermitian metric, and in particular, for a speical case, we take the base metric as *Kähler*. Now, the structure of this paper is as follows. Chapter 2 builds solid fundamental geometric aspects of complex geoemtry, starting with complex vector bundles, connections, and curvature. The main reference is from chapter 1, *Differential Geometry of Vector Bundles* by Shoshichi Kobayashi [6], and it is explained in a sophisticated manner. Next, analyzing the differential forms allow us to represent deRham cohomology class and this is chapter 3 on Characteristic classes. We focus on the Second Bianchi Identity and the Chern-Weil theory; for the reference, it is heavily from section 13,14, and appendix C from *Characteristic Classes* by Milnor and Stasheff [11]. The following chapter 4 thoroughly demonstrates the core of this paper. We state the main theorem.

Theorem 1.2.1 (Donaldson [3],Yang-Mills [14]). *The Yang-Mills functional is the following functional associated with any unitary connection, ∇, such that,*

$$\nabla \to \mathcal{I}(\nabla) = \int_{\mathcal{X}} \|\mathcal{F}_{\nabla}\|^2 \sqrt{g}\, dx \tag{1.1}$$

Then, the critical points of the Yang-Mills functional $\mathcal{I}(\nabla)$ is the Yang-Mills equation

$$\nabla^{\mathfrak{p}} \mathcal{F}_{\mathfrak{p}\mathfrak{q}}{}^{\alpha}_{\beta} = 0 \tag{1.2}$$

,and rigorously by the variational formula, we can express the Yang-Mills equation as

$$d_{\nabla}^{\dagger} \mathcal{F} = 0 \tag{1.3}$$

Then, in chapter 5, we show the basic examples and solutions of the *Yang-Mills theory*. Start with the *Maxwell's equation* [8] as an example, then (anti)self-dual solutions. In fact, the author collected concrete celebrated result from *Anti Self-Dual Yang-Mills Connections over Complex Algebraic Surfaces and Stable Vector Bundles* by Simon Donaldson [2]. Stretching from the self-dual solutions, Belavin-Polyakov-Schwartz-Tyuptin demonstrated solutions known as the *BPST-Instantons* [1], and this is on the last section of chapter 5. The final chapter of this paper explains the *Yang-Mills theory* on *Kähler* case. The basic introduction of *Kähler* geometry is from chapter 11,13, and 13 from *Lecture on Kähler Geometry* by Moroianu [9] and section 1 from *Holomorphic Vector Fields on Compact Kähler Manifolds* by Matshshima [7]. The *Yang-Mills equation* on *Kähler* manifolds is from Donaldson [2] and Uhlenbeck-Yau [12].

1.3 Fundamental Concepts

We first begin by unpacking the original definitions from the article by R.L. Mills and C.N. Yang.

Definition 1.3.1 (Gauge invariance, Nakahara [10]). If the Lagrangian(i.e.action) from the field theory remains unchanged, then it is said to be gauge invariant. Formally, if we have a field ψ, associated with a local transformation T belonging to its Lie group(i.e.$T \in Lie(\psi)$), then the transformation is followed as

$$\psi \to \psi' = T \cdot \psi$$

, where T is depend on unitiary space-time metric. e.g. $det(T) = 1$ in $SU(2)$. Since $\partial_j \psi$ would not covariantly transformed in local coordinates, we introduce a connection $\mathcal{A}_j$(i.e.Gauge potential), and define the covariant derivative by

$$\nabla_j \psi = \partial_j \psi + ig\mathcal{A}_j \psi$$

Note that g is a gauge coupling constant, which quantifies the strength of the interaction between the field ψ and the connection $\mathcal{A}_j$.

Remark 1.3.2 (Kirillov [5]). A Lie group(real or complex) is a smooth manifold with a group structure given by a multiplication map $m: G \times G \to G$ and an inversion $i: G \to G$, such that, both m and i are smooth. For example, we have

$$SU(2) = \{A \in GL(2,\mathbb{C}) | A\bar{A}^t = 1, det(A) = 1\}$$

, and explicitly, we can define

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

If we take $\alpha = x_1 + ix_2$ and $\beta = x_3 + ix_4$ for any $x_i \in \mathbb{R}$, then we easily see that $SU(2)$ is diffeomorphic to $S^3 \subset \mathbb{R}^4$.

Definition 1.3.3 (Isotopic gauge, Yang-Mills [14]). The essence of isotopic spin is the idea that neutrons and protons are not distinct, but two states of the same particle(i.e.field). Let φ be a field that carries isotopic spin with local transformation $T \in SU(2)$. Then an isotopic spin rotation is the local $SU(2)$ gauge transformation followed by the gauge invariance

$$\varphi \to \varphi' = T \cdot \varphi$$

By the same analogue from gauge invariance, given the connection $\mathcal{A}$, we define the covariant derivative as $\nabla_j \varphi = \partial_j \varphi + ig\mathcal{A}_j \varphi$. The covariant derivative transforms homogenously

$$\nabla_j \varphi \to \nabla_j{}' \varphi = T \nabla_j \varphi$$

,where the associated curvature of the gauge field is given by

$$\mathcal{F}_{jk} = \partial_j \mathcal{A}_k - \partial_k \mathcal{A}_j + ig[\mathcal{A}_j, \mathcal{A}_k]$$

Having said, the isotopic gauge is arbitrary way of choosing the orientiation of the isotopic spin axes at all space time, extending the notion of local gauge invariance with respect to $U(1)$ symmetry.

Establishing the basic notions of complex analysis and geometry is fruitful since this will be the main language of what follows. Though, of course, such detailed proofs will be omitted as complex analysis itself is not our focus. Further definitions below are straight from chapter 1,2, and 3 of *Complex Analysis* by Stein and Shakarachi [11], and chapter 1 and 2 of *Complex Geometry An Introduction* by Huybrechts [4].

Definition 1.3.4 (Holomorphic and meromorphic functions). Let $U \to \mathbb{C}$ be and open subset. A function $f : U \to \mathbb{C}$ is called *holomorphic*, if there exists a ball $B_\epsilon(z_j)$ of radius $\epsilon > 0$ around for any point $z_j \in U$, such that the function f can be written as a convergent power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_j)^n \text{ for all } z \in B_\epsilon(z_j)$$

,or equivalently, a function f is said to be *holomorphic*, if for any point $z_j \in U$ is complex differentiable. Which means, the limit

$$f'(z_j) = \lim_{h \to 0} \frac{f(z_j + h) - f(z_j)}{h} \text{ exists for all } z_j \in U$$

The intuition of *holomorphicity* encourages from realizing the condition of complex differentiability and hence analytic in any open subset of $\mathbb{C}$. Next, a function f is *meromorphic* in U, if there exsits a sequence of points $\{z_0, z_1, z_2, \ldots\}$ that has no limit poins in U, and such that the f is *holomorphic* in $U \setminus \{z_0, z_1, z_2, \ldots\}$, and f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

Remark 1.3.5. A complex number z_j is *zero* for the holomorphic function f is $f(z_j) = 0$. A deleted neighborhood of z_j is denoted by the set $\mathcal{O} = \{z : 0 < |z - z_j| < r\}$ for some $r > 0$. Then, a function f defined on $\mathcal{O}$ has a *pole* at z_j. Conversely, if the funtion $\frac{1}{f}$ is *zero* at z_j, it is *holomorphic* in a full neighborhood of z_j.

Definition 1.3.6 (Cauchy-Riemann equations). The complex function f of two variables can be written in the form $f(x,y) = \mu(x,y) + i\nu(x,y)$. Then f is holomorphic if and only if μ, ν are continuously differentiable and satisfying the equation

$$\frac{\partial \mu}{\partial_x} = \frac{\partial \nu}{\partial y}, \ \frac{\partial \mu}{\partial_y} = -\frac{\partial \nu}{\partial x}$$

Definition 1.3.7 (Complex and Hermitian structure)**.** Let V be a finite dimensional real vector space. Then, an endomorphism $J:V\to V$ with $J^2=-Id$ is called *(almost) complex structre* on V. This map J must satisfies the linearity, such that, for all $x,y\in\mathbb{R}$ and $u,v\in V$,

$$J(xv+yu)=xJv+yJu$$

Now a positive-definite bilinear form $H:V\times V\to\mathbb{R}$, such that, for all $u,v\in V$,

$$H(Ju,Jv)=H(u,v)$$

is called *Hermitian structure* on V. It is often described as a complex inner product on complex vector space.

Definition 1.3.8 (Complex manifold)**.** A *complex manifold* $\mathcal{X}$ of $\mathbb{C}^n$ is a topological space, satisfying Hausdorffness, second countability, and locally euclidean. That is, from a *holomorphic atlas* (U_j,φ_j) of the form $\varphi_j:U_j\simeq\varphi_j(U_i)\subset\mathbb{C}^n$, the transition functions

$$\varphi_{jk}=\varphi_j\circ{\varphi_k}^{-1}:\varphi_k(U_j\cap U_k)\to\varphi_j(U_j\cap U_k)$$

are holomorphic. So, a n-dimensional *complex manifold* $\mathcal{X}$ is a 2n-dimensional real differentiable manifold endowed with an equivalence class of holomorphic atlases.

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Chapter 2: Complex Vector Bundles, Connections, and Curvature

2.1 Vector Bundles

Complex vector bundles are smooth vector bundles, whose fibers are complex vector spaces. Initially, we can consider complex vector bundles in which each fiber is a vector space over the complex numbers. Also, since the notion of smooth vector bundles is defined on smooth manifolds, we fix topological spaces M and E be smooth manifolds. Then we can apply the condition of *local triviality*: for each point $m \in M$, there exists a neighborhood $U \subset M$, and a diffeomorphism

$$\varphi : U \times \mathbb{C}^n \to \pi^{-1}(U) \text{ for any } n \in \mathbb{Z}_{\geq 0}$$

where $\pi : E \to M$ is a smooth projection map. i.e. For each $m \in M$, there exists a pair (U, φ), which is a local coordinate system with $m \in U$, such that φ be a diffeomorphism. Here, the vector space $\pi^{-1}(m)$ is called the *fiber* over m. Satisfying the *local triviality requirements*, we obtain complex vector bundles. However, for a geometric intuition, we are especially interested in bundles on *Riemann Surfaces*, a one-dimensional complex manifolds. Here, we can observe the concrete structre, which is a line bundle, where the fibers are one-dimensional complex vector space. Proceeding it further, we will define *line bundles*, in order to construct *vector bundles*.

Definition 2.1.1 (Riemann Surface)**.** $\mathcal{X}$ is called a *Riemann Surface*, if $\mathcal{X} = \bigcup X_\mu$(union of small neighborhoods) defined on a complex plane $\mathbb{C}$, having the property

$$\Phi_\mu : X_\mu \to \mathcal{D}(\text{unit disk}) \subseteq \mathbb{C}$$

, and for all μ, ν the local coordiate map:

$$\Phi_\nu \circ \Phi_\mu^{-1} : \Phi_\mu(X_\mu \cap X_\nu) \to \Phi_\nu(X_\mu \cap X_\nu)$$

is holomorphic, 1-1 form, and $(\Phi_\nu \circ \Phi_\mu^{-1})'(z) \neq 0$(invertible differential). Note that for a notational sanity, z is coordinate of the point in X_μ, which is $\Phi_\mu : z \mapsto z_\mu \in \mathcal{D}$.

Definition 2.1.2. A holomorphic *line bundle* $\mathcal{L} \to \mathcal{X}$, denoted $\mathcal{L}$ over $\mathcal{X}$ is an assignment

$$\mathcal{L} \longleftrightarrow \{t_{\mu\nu}(z) \,\text{characterized by transition functions on}\, X_\mu \cap X_\nu\}, \text{satisfying}$$

$$\text{the cocyle condition:}\, t_{\mu\nu} t_{\nu\xi} = t_{\mu\xi} \,\text{on}\, X_\mu \cap X_\nu \cap X_\xi$$

Here, we have $\Gamma(\mathcal{X}, \mathcal{L}) = \{\text{sections}\, \varphi \,\text{of}\, \mathcal{L}\}$, and in practice, we view *line bunde* $\mathcal{L}$ as same as the space $\Gamma(\mathcal{X}, \mathcal{L})$ of its sections, so then

$$\Gamma(\mathcal{X}, \mathcal{L}) \ni \varphi \longleftrightarrow \varphi_\mu(z_\mu) \,\text{functions on}\, \Phi_\mu(X_\mu), \text{satisfying}$$

$$\text{the glueing condition:}\, \varphi_\mu(z_\mu) = t_{\mu\nu}(z) \varphi_\nu(z_\nu) \,\text{on}\, X_\mu \cap X_\nu$$

Definition 2.1.3. Since the set up is similar to line bundles, a holomorphic *vector bundle* $\mathcal{E} \to \mathcal{X}$, denoted $\mathcal{E}$ over $\mathcal{X}$, is an assignment

$$\mathcal{E} \longleftrightarrow \{t_{\mu\nu}{}^\alpha_\beta, 1 \leq \alpha, \beta \leq r, \text{defined on}\, X_\mu \cap X_\nu\}, \text{satisfying}$$

$$\text{the cocyle condition:} \, t_{\mu\nu}{}^{\alpha}_{\beta} t_{\nu\xi}{}^{\beta}_{\gamma} = t_{\mu\xi}{}^{\alpha}_{\gamma} \, \text{defined on} \, X_\mu \cap X_\nu \cap X_\xi$$

Note that such trasition functions here are matrix-valued functions, having two indices α and β. Here, we also have smooth sections: $\Gamma(\mathcal{X},\mathcal{E}) = \{\text{sections of} \, \varphi \, \text{of} \, \mathcal{E}\}$, and see

$$\Gamma(\mathcal{X},\mathcal{E}) \ni \varphi \longleftrightarrow {\varphi_\mu}^\alpha(z_\mu) \, \text{vector-valued funtions on} \, \Phi_\mu(X_\mu), \text{satisfying}$$

$$\text{the glueing condition:} \, {\varphi_\mu}^\alpha(z_\mu) = t_{\mu\nu}{}^{\alpha}_{\beta}(z) \varphi^\beta_\nu(z_\nu) \, \text{on} \, X_\mu \cap X_\nu$$

Now, from the sections, we fix a holomorphic vector bundle $\Gamma(\mathcal{X},\mathcal{E}) \ni \varphi = \{\varphi_\mu\}$. Then the covariant derivatives are defined as:

$$\bar{\partial}\varphi \triangleq \{\frac{\partial}{\partial \bar{z}^j} \varphi d\bar{z}^j\} = \{\frac{\partial}{\partial \bar{z}^j}_\mu \varphi^\alpha_\mu d\bar{z}^j_\mu \, \text{on} \, X_\mu\}$$

Since the glueing condition is $\varphi^\alpha_\mu = t_{\mu\nu}{}^{\alpha}_{\beta} \varphi^\beta_\nu$ on $X_\mu \cap X_\nu$, we can easily check that

$$\frac{\partial}{\partial \bar{z}^j}_\mu \varphi^\alpha_\mu = t_{\mu\nu}{}^{\alpha}_{\beta} \frac{\partial}{\partial \bar{z}^j}_\mu \varphi^\beta_\nu = t_{\mu\nu}{}^{\alpha}_{\beta} \frac{\partial \bar{z}^k_\nu}{\partial \bar{z}^j_\mu} \frac{\partial}{\partial \bar{z}^k_\nu} \varphi^\beta_\nu = t_{\mu\nu}{}^{\alpha}_{\beta} \left(\frac{\partial}{\partial \bar{z}^k_\nu} \varphi^\beta_\nu \right) \frac{\partial}{\partial \bar{z}^k_\nu}$$

, and notice that such naive differentiation of φ^α_μ would only result in vector-valued funtions, but not satisfying desired glueing condition. In order to differentiate in z^j direction, we have to introduce *connections*.

2.2 Connections

Definition 2.2.1. A *connection* on $\mathcal{E}$ is a linear mapping, such that,

$$\nabla : \Gamma(\mathcal{X},\mathcal{E}) \to \Gamma(\mathcal{X},\mathcal{E} \otimes \Lambda^1), \, \text{where} \, \Lambda^1 \, \text{is a bundle of one-form on} \, \mathcal{X}.$$

In fact, this can be described locally as $\nabla \varphi^\alpha = dx^j \nabla_j \varphi^\alpha$ with $\nabla_j \varphi^\alpha = \partial_j \varphi^\alpha + A^\alpha_{j\;\beta} \varphi^\beta$.

One significant connection is *Chern Unitary Connection*. To see this, we will define *Chern connection* and *unitary condition*.

Definition 2.2.2 (Chern connections)**.** Fix a holomorphic vector bundle $\mathcal{E} \to \mathcal{X}$, where $\mathcal{X}$ is a complex manifold. i.e. The coordinate map $\Phi_\nu \circ \Phi_\mu^{-1}$ is a holomorphic function, having the glueing condition:

$${\varphi_\mu}^\alpha(z_\mu) = t_{\mu\nu}{}^{\alpha}_{\beta}(z) \varphi^\beta_\nu(z_\nu) \, \text{defined on} \, X_\mu \cap X_\nu$$

with a holomorphic trasition function $t_{\mu\nu}{}^{\alpha}_{\beta}(z)$. Here, if we take $z_\mu = {z_\mu}^k$, for $1 \leq k \leq dimX$, then, by the holomorphicity of $t_{\mu\nu}{}^{\alpha}_{\beta}(z)$, we have:

$$\frac{\partial}{\partial \bar{z}^k_\mu} \varphi^\alpha_\mu(z_\mu) = t_{\mu\nu}{}^{\alpha}_{\beta}(z) \frac{\partial}{\partial \bar{z}^k_\mu} {\varphi_\nu}^\beta(z_\nu) = t_{\mu\nu}{}^{\alpha}_{\beta}(z) \frac{\partial \bar{z}^j_\nu}{\partial \bar{z}^k_\mu} \frac{\partial}{\partial \bar{z}^j_\nu} \varphi^\beta_\nu(z_\nu)$$

This means, $\frac{\partial}{\partial \bar{z}^k_\mu} \varphi^\alpha_\mu(z_\mu)$ transforms as a section of a vector bundle with a transition function $t_{\mu\nu}{}^{\alpha}_{\beta}(z) \frac{\partial \bar{z}^j_\nu}{\partial \bar{z}^k_\mu}$; in fact, we may see that $t_{\mu\nu}{}^{\alpha}_{\beta}(z)$ as $\mathcal{E}$, and $\frac{\partial \bar{z}^j_\nu}{\partial \bar{z}^k_\mu}$ as $\Lambda^{0,1}-$form. i.e. This is

$$\frac{\partial}{\partial \bar{z}^k_\mu} \varphi^\alpha_\mu(z_\mu) \in \Gamma(\mathcal{X},\mathcal{E} \otimes \Lambda^{0,1})$$

, and we can always define the *Chern connection* $\nabla_{\bar{k}} \varphi^\alpha$ as

$$\Gamma(\mathcal{X},\mathcal{E}) \ni \varphi \to \nabla_{\bar{k}} \varphi^\alpha = \frac{\partial \varphi^\alpha}{\partial \bar{z}_k} \in \Gamma(\mathcal{X},\mathcal{E} \otimes \Lambda^{0,1})$$

Now, the holomorphicity of a vector bundle $\mathcal{E} \to \mathcal{X}$ determines only half of the connection: $\nabla_{\bar{k}}$, which we defined above. In order to show ∇_j, we need such *unitary* of a connection, and we will proceed further to define the unitary condition.

Definition 2.2.3 (The Unitary Condition)**.** Here, we fix a metric on $\mathcal{E}$ to be Hermitian, denoted $H_{\bar{\alpha}\beta}$ on $\mathcal{E}$. The *Hermitian metric* $H_{\bar{\alpha}\beta}$ determines by the inner product on $\Gamma(\mathcal{X},\mathcal{E})$:

$$\Gamma(\mathcal{X},\mathcal{E}) \ni \varphi^\alpha, \psi \to \langle \varphi, \psi \rangle = \varphi^\alpha \overline{\psi^\beta} H_{\bar{\beta}\alpha}, \text{ which is a scalar.}$$

Then, we say a *connection* ∇ is *unitary*, if

$$\partial_j \langle \varphi, \psi \rangle = \langle \nabla_j \varphi, \psi \rangle + \langle \varphi, \nabla_{\bar{k}} \psi \rangle$$

Hence, having the *unitary condition*, we can solve for $\nabla_j \varphi$:

$$\partial_j (\varphi^\alpha \overline{\psi^\beta} H_{\bar{\beta}\alpha}) = (\nabla_j \varphi)^\alpha H_{\bar{\beta}\alpha} \overline{\psi^\beta} + \varphi^\alpha \overline{(\partial_{\bar{k}} \psi)^\beta} H_{\bar{\beta}\alpha}$$

; so then,

$$\overline{\psi^\beta} (\partial_j \varphi^\alpha) H_{\bar{\beta}\alpha} + \varphi^\alpha \overline{\psi^\beta} \partial_j H_{\bar{\beta}\alpha} = (\nabla_j \varphi)^\alpha H_{\bar{\beta}\alpha} \overline{\psi^\beta}$$

,and since

$$(\partial_j \varphi^\alpha) H_{\bar{\beta}\alpha} + \varphi^\alpha \partial_j H_{\bar{\beta}\alpha} = (\nabla_j \varphi)^\alpha H_{\bar{\beta}\alpha}$$

we find

$$(\nabla_j \varphi)^\alpha = H^{\alpha\bar{\beta}} \partial_j \left(H_{\bar{\beta}\gamma} \varphi^\gamma \right), \text{ where } \left(H_{\bar{\beta}\gamma} \varphi^\gamma \right) \text{ is anti-holomorphic bundle.}$$

Definition 2.2.4. Putting *Chern connection* and *the unitary condition*, the following is well-defined connection in a basic complex geometry:

$$\begin{cases} \Gamma(\mathcal{X},\mathcal{E}) \ni \varphi & \to \nabla_{\bar{k}} \varphi^\alpha \in \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{0,1}) \\ \Gamma(\mathcal{X},\mathcal{E}) \ni \varphi & \to \nabla_j \varphi^\alpha = H^{\alpha\bar{\beta}} \partial_j \left(H_{\bar{\beta}\gamma} \varphi^\gamma \right) \end{cases}$$

, and this is called the *Chern unitary connection*.

2.3 Curvature

Knowing the connection and vector bundles, we are now ready to show the curvature, and to do this, we first introduce exterior differentials.

Definition 2.3.1. The extented differential is defines as:

$$d_\nabla : \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}}) \to \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}+1})$$

Recall that on any smooth manifold $\mathcal{X}$, there is a well-defined exterior differential operator

$$d : \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}}) \to \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}+1}) \text{ characterized by } df = dx^j \frac{\partial f}{\partial x^j} \text{ for any } f \in \Gamma(\mathcal{X}, \Lambda^0)$$

However, such definitions do not require any further structure;so, what we need to focus in topology is the following the equation:

$$d^2 = 0 \tag{2.1}$$

Suppose we are give a vector bundle $\mathcal{E} \to \mathcal{X}$, then we can generalize the exterior differential operator:

$$d : \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}}) \to \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}+1}) \text{ to an operator } d_\nabla : \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}}) \to \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{\mathfrak{p}+1})$$

by setting naively

$$d_\nabla\big(\sum_{|J|=\mathfrak{p}}\varphi^\alpha_J dx^J\big)=\sum_{|J|=\mathfrak{p}}(d_\nabla\varphi^\alpha_J)\wedge dx^J\ (\Lambda^{\mathfrak{p}+1}\text{ valued form in }\mathcal{E})$$

Knowing the connection∇, it is easy to check that d_∇ is well-defined; however, the equation $d^2=0$ is now modifed as the following:

$$d^2_\nabla\varphi=\mathcal{F}\wedge\varphi\text{ for }\varphi\in\Gamma(\mathcal{X},\mathcal{E}\otimes\Lambda^{\mathfrak{p}}),\text{ where }\mathcal{F}\text{ is vector-valued two forms}\tag{2.2}$$

To see this, let $\varphi\in\Gamma(\mathcal{X},\mathcal{E})$, we compute:

$$\begin{aligned}
d_\nabla\varphi&=\nabla_j\varphi^\alpha dx^j\\
d^2_\nabla\varphi&=d_\nabla(\nabla_j\varphi^\alpha)\wedge dx^j\\
&=\nabla_k(\nabla_j\varphi^\alpha)dx^k\wedge dx^j\ \text{(anti-symmetrized)}\\
&=\frac{1}{2}(\nabla_k\nabla_j\varphi^\alpha-\nabla_j\nabla_k\varphi^\alpha)dx^k\wedge dx^j\ \text{(by commutator)}\\
&=\frac{1}{2}\left(\mathcal{F}_{kj}{}^\alpha_\beta\varphi^\beta\right)dx^k\wedge dx^j
\end{aligned}$$

Now, close work on the commutator, where

$$\nabla_k\nabla_j\varphi^\alpha-\nabla_j\nabla_k\varphi^\alpha=\nabla_k(\partial_j\varphi^\alpha+\mathcal{A}_j{}^\alpha_\beta\varphi^\beta)-\nabla_j(\partial_k\varphi^\alpha+\mathcal{A}_k{}^\alpha_\beta\varphi^\beta)$$

expand RHS further to see

$$=\partial_k\left(\partial_j\varphi^\alpha+\mathcal{A}_j{}^\alpha_\beta\varphi^\beta\right)+\mathcal{A}^\alpha_{k\gamma}\left(\partial_j\varphi^\gamma+\mathcal{A}_j{}^\gamma_\beta\varphi^\beta\right)-\partial_j\left(\partial_k\varphi^\alpha+\mathcal{A}_k{}^\alpha_\beta\varphi^\beta\right)-\mathcal{A}_j{}^\alpha_\gamma\left(\partial_k\varphi^\gamma+\mathcal{A}_k{}^\gamma_\beta\varphi^\beta\right)=\mathcal{F}_{kj}{}^\alpha_\beta\varphi^\beta$$

with, $\mathcal{F}_{kj}{}^\alpha_\beta\varphi^\beta=\partial_k\mathcal{A}^\alpha_{j\ \beta}-\partial_j\mathcal{A}^\alpha_{k\,\beta}+\mathcal{A}^\alpha_{k\,\gamma}\mathcal{A}^\gamma_{j\,\beta}-\mathcal{A}^\alpha_{j\,\gamma}\mathcal{A}^\gamma_k$.

Clearly, in matrix notation,

$$\mathcal{F}_{kj}=\partial_k\mathcal{A}_j-\partial_j\mathcal{A}_k+\mathcal{A}_k\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_k$$

In summary, the curvature of complex vector bundles is given by

$$[\nabla_j,\nabla_k]\varphi^\alpha=\mathcal{F}_{\bar{k}j}{}^\alpha_\beta\varphi^\beta\text{ , where }\mathcal{F}_{\bar{k}j}{}^\alpha_\beta=-\partial_{\bar{k}}\mathcal{A}_j{}^\alpha_\beta\text{ , and}$$

$$\mathcal{F}=\mathcal{F}^\alpha_{\bar{k}j\,\beta}dx^j\wedge d\bar{x}^k\in\Gamma(\mathcal{X},\Lambda^{1,1}\otimes End(\mathcal{E}))$$

More generally, for any connection $\mathcal{A}$, the curvature is defined with the form

$$\mathcal{F}=d\mathcal{A}+\mathcal{A}\wedge\mathcal{A}$$

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Chapter 3: Characteristic Classes

In this chapter, we set $\mathcal{E} \to \mathcal{X}$ be complex vector bundles as we defined above. The *characteristic classes* of complex vector bundles are *Chern class*. From the PDE point of view, our curvature $\mathcal{F}_{\mathcal{A}}$ satisfies a set of non-linear PDEs. Then we first introduce the Bianchi identity as it ensures the consistency and well-defineness of the curvature form.

3.1 Bianchi Identity

Definition 3.1.1 (The Second Bianchi Identity)**.** We first recall that $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. So then, by the Leibniz' rule,

$$\begin{aligned} d\mathcal{F} &= d(\mathcal{A} \wedge \mathcal{A}) = d\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge d\mathcal{A} \\ &= (d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}) - \mathcal{A} \wedge (d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}) \\ &= \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} \end{aligned}$$

Then, we find the identity:

$$d\mathcal{F} - \mathcal{F} \wedge \mathcal{A} + \mathcal{A} \wedge \mathcal{F} = 0 \tag{3.1}$$

It is important to note that any curvature always satisfies this second Bianchi identity. Next, as we view the curvature $\mathcal{F}$ in a section of $\Lambda^2 \otimes End(\mathcal{E})$, we may observe that the exterior differential $d_\nabla \mathcal{F} = d\mathcal{F} + \mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A} = 0$. Thus, we find

$$\text{The Second Bianchi Identity holds if and only if } d_\nabla \mathcal{F} = 0 \tag{3.2}$$

3.2 Chern Class

In this section, we will show the key theorems of the *Chern-Weil Theory*. In fact, we stated complex vector bundles by showing the line bundles; so, following the natural analogy, we first show the Chern-class from holomorphic line bundles over Riemann surfaces.

Definition 3.2.1. Let $\mathcal{L} \to \mathcal{X}$ be holomorphic lines bundles over Riemann surface $\mathcal{X}$. Then, a metric H on $\mathcal{L}$ is a section φ of $\mathcal{L}^{-1} \otimes \bar{\mathcal{L}}^{-1}$, which satisfies $H > 0$ so that

$$\Gamma(\mathcal{X}, \mathcal{L}) \ni \varphi \to H_{\varphi\bar{\varphi}} = |\varphi|^2 > 0 \text{ which is a positive scalar}$$

Then the Chern-unitary connection is given by

$$\nabla_z \varphi = H^{-1} \partial_z (H_\varphi) = \partial_z \varphi + \Big(H^{-1} \partial_z H\Big) \varphi \text{ and } \nabla_{\bar{z}} \varphi = \partial_{\bar{z}} \varphi$$

Since $H^{-1} \partial_z H = \mathcal{A}_z$, in matrix notation, we can express this as

$$\mathcal{A}_{\bar{z}} = 0, \mathcal{A}_z = \partial_z \, logH$$

The curvature of $\mathcal{L}$ is shown by the commutator rule

$$\begin{aligned} [\nabla_{\bar{z}}, \nabla_z]_\varphi &= (\nabla_{\bar{z}} \nabla_z) \varphi - (\nabla_z \nabla_{\bar{z}}) \varphi = \partial_{\bar{z}} \{\partial_z \varphi + (\partial_{\bar{z}} logH) \varphi\} - \{\partial_j (\partial_{\bar{z}} \varphi) + (\partial_z logH) \partial_{\bar{z}} \varphi\} \\ &= (\partial_{\bar{z}} \partial_z logH) \varphi = -\mathcal{F}_{\bar{z}z} \varphi \\ i.e. \mathcal{F}_{\bar{z}z} &= -\partial_{\bar{z}} \partial_z logH \text{ and the curvature form } \mathcal{F} = \mathcal{F}_{\bar{z}z} dz \wedge d\bar{z} \end{aligned}$$

Finally, we end up with defining the *first Chern-class*. Consider $\Gamma(\mathcal{X},\mathcal{L}) \ni \varphi$ be meromorphic, then we have

$$(\sharp\text{zeroes of } \varphi) - (\sharp\text{poles of } \varphi) = \frac{i}{2\pi}\int_{\mathcal{X}} \mathcal{F}_{z\bar{z}} dz \wedge d\bar{z} = \frac{i}{2\pi}\int_{X} \mathcal{F} = C_1(\mathcal{L})$$

Now, let $\mathcal{E} \to \mathcal{X}$ be a smooth complex vector bundle. For any connection ∇ having the matrix form $\mathcal{A}$ and curvature form $\mathcal{F}$, we set a rank for any integer $i \in \mathbb{Z}$.

$$C_i(\mathcal{A}) = Tr(\mathcal{F} \wedge \cdots (i - factors) \cdots \wedge \mathcal{F}) \in \Gamma(\mathcal{X}, \Lambda^{2i})\text{ , where } \mathcal{F} \text{ is in a space of endomorphism}$$

Theorem 3.2.2. *$C_i(\mathcal{A})$ is always a closed form*

Proof. We employ the second Bianchi identity and impose Leibniz rule. Then,

$$\begin{aligned} dC_i &= Tr\sum(\mathcal{F} \wedge \cdots \wedge d\mathcal{F} \wedge \mathcal{F}) \\ &= Tr\sum(F \wedge \cdots \wedge (-\mathcal{A} \wedge \mathcal{F} + \mathcal{F} \wedge \mathcal{A}) \wedge \cdots \mathcal{F}) \\ &= 0, \text{since}\, Tr(AB) = Tr(BA) \;\text{ which is commutative.} \end{aligned}$$

Q.E.D.

Theorem 3.2.3. *The deRham cohomology class does not depend on the connection $\mathcal{A}$. We call $[C_i(\mathcal{A})]$ be the i-th characteristic class of a vector bundle $\mathcal{E}$.*

Proof. Let $\mathcal{A}$,$\mathcal{A}'$ be two connections. Fix B be a well-defined one-form, and write $\mathcal{A}' = \mathcal{A} + B$. We claim that $C_i(\mathcal{A}') - C_i(\mathcal{A}) = dT_i$, where T_i is (i-1)-form explicitly given by

$$T_i = m\int_0^1 Tr(B \wedge {F_t}^{i-1})dt \,\text{where } F_t \text{ curvature of the connection}\, \mathcal{A}_t = \mathcal{A} + tB$$

Note that $\dot{\mathcal{A}}_t = \frac{d}{dt}\mathcal{A}_t$. In fact,

$$\dot{\mathcal{F}}_t = d\dot{\mathcal{A}}_t + \dot{\mathcal{A}}_t \wedge \mathcal{A}_t + \mathcal{A}_t \wedge \dot{\mathcal{A}}_t = dB + B \wedge \mathcal{A}_t + B \wedge \mathcal{A}_t + \mathcal{A}_t \wedge B$$

This implies

$$\begin{aligned} C_i(\mathcal{A}') - C_i(\mathcal{A}) &= \int_0^1 \frac{d}{dt} C_i(\mathcal{A}_t)dt \\ &= \int_0^1 \sum Tr(\mathcal{F}_t \wedge \dot{\mathcal{F}}_t \wedge \cdots \wedge \mathcal{F}_t)dt \\ &= i\int_0^1 \sum Tr(\dot{\mathcal{F}}_t \wedge {\mathcal{F}_t}^{i-1})dt \\ &= i\int_0^1 \sum Tr(dB + B \wedge \mathcal{A}_t + \mathcal{A}_t \wedge B) \wedge \mathcal{F}_t \end{aligned}$$

We verify this by using the second Bianchi identity, which states that:

$$d_{\nabla}\mathcal{F}_t = d\mathcal{F}_t + \mathcal{A}_t \wedge \mathcal{F}_t - \mathcal{F}_t \wedge \mathcal{A}_t = 0$$

First multiplyting ${\mathcal{F}_t}^{i-1}$ and taking the trace to get

$$Tr({\mathcal{F}_t}^{i-1} \wedge d\mathcal{F}_t) = -Tr({\mathcal{F}_t}^{i-1} \wedge (\mathcal{A}_t \wedge \mathcal{F}_t - \mathcal{F}_t \wedge \mathcal{A}_t))$$

Since we are integrating over dB, so we expand this as

$$d_{\nabla}B = dB + \mathcal{A}_t \wedge B - B \wedge \mathcal{A}_t$$

,and rewrite our integral to

$$C_i(\mathcal{A}') - C_i(\mathcal{A}) = i\int_0^1 \sum Tr(d_{\nabla}B \wedge {\mathcal{F}_t}^{i-1})dt$$

Notice that the exterior derivative commutes with the integration, we find

$$C_i(\mathcal{A}') - C_i(\mathcal{A}) = d\left(i\int_0^1 \sum Tr(B \wedge {\mathcal{F}_t}^{i-1})dt\right)$$

Thus, we obtain

$$C_i(\mathcal{A}') - C_i(\mathcal{A}) = dT_i$$

, and hence, we have shown that $[C_i(\mathcal{A})]_{dR}$ is independent of connection ∇ and $\mathcal{A}$. Q.E.D.

Chapter 4: The Yang-Mills Functional, Its Variations, and The Yang-Mills Equation

For a general condition, let $\mathcal{E} \to \mathcal{X}$ be a smooth complex vector bundle over a C^∞ manifold $\mathcal{X}$, equipped with a Hermitian metric $H_{\bar{\alpha}\beta}$. We also set g_{ij} be a C^∞ metric on $\mathcal{X}$.

4.1 The Yang Mills Functional

The *Yang-Mills functional* is the following functional associated with any unitary connection with respect to the given metric $H_{\bar{\alpha}\beta}$:

$$\nabla \to \mathcal{I}(\nabla) = \int_{\mathcal{X}} \|\mathcal{F}_\nabla\|^2 \sqrt{g}dx \tag{4.1}$$

defined over all unitary connections ∇ (we could also write as a metric form A). Note that $\|\mathcal{F}_\nabla\|^2$ is L^2 norm of the curvature with the connection ∇ and $\sqrt{g}dx$ is the volume form associated with the metric g. i.e. Given the connection ∇, we observe the curvature F_∇, and clearly, this is a 2-2 form bilinear space of endomorphism. Since the unitary connection could be written as a metric from $\mathcal{A}$, we see the curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ be the curvature of ∇. For a disclaimer, this function governs toward high-energy fundamental laws of nuclear physics. Historically, from a geometric pespective in 1950s, certain geometries came up with fibre bundles, connections, and curvature, but yet to find this functional. In fact, as the one understands the *Yang-Mills functional* and *equation*, it connects the mathematics and physics in turn. Now, our interests is in the *critical points* of this functional, known as the *Yang-Mills Equation*.

4.2 Basic Approach - A First Glance

From advanced calculus, we know that critical points are defined as the points where the derivative of such functional is zero. More generally, a connection ∇ is considered to be a critical point if there exists no variations that change the value of the functional $\mathcal{I}(\nabla)$. In this sense, the Yang-Mills equation is the Euler-Lagrange equation for $\mathcal{I}(\nabla)$. For instance, if we take the volume of the functional, then the critical points would simply give us the minimal area of the submanifold. In order to derive it, we will apply the following sequence of the variations: (1)variations of the unitary connection ∇, (2)variations of the curvature, and (3)functional variations. Explicitly, for a computation wise, we use this:

1. Basic set up: Let the unitary connection 1-form A on a principal vector bundle $\mathcal{E} \to \mathcal{X}$. Locally, we can write as $\mathcal{A} = \mathcal{A}_k dx^k$, where each $\mathcal{A}_k$ is generally $End(\mathcal{E})$-valued function. i.e. We can see that it takes the values in the endomorphism of the fiber.

2. Curvature F: The curvature 2-form $\mathcal{F}$, denoted $\mathcal{F}_{\mathfrak{pq}}$ of this connection is given by $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. In local coordinates, we have $\mathcal{F} = \frac{1}{2}\mathcal{F}_{jk} dx^j \wedge dx^k$ where $\mathcal{F}_{jk} = \partial_j \mathcal{A}_k - \partial_k \mathcal{A}_j + [\mathcal{A}_j, \mathcal{A}_k]$.

3. Variation $\delta\mathcal{A}$: We consider a small perturbation of $\mathcal{A}$, such that, $\mathcal{A} \mapsto \mathcal{A} + \delta\mathcal{A}$, with $\delta\mathcal{A} = \delta\mathcal{A}_k dx^k$. Our goal now is to compute the resulting variation $\delta\mathcal{A}$ of the curvature.

Proceeding it further, we start from $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$, and the variation is

$$\delta\mathcal{F} = \delta(d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}) = \delta(d\mathcal{A}) + \delta(\mathcal{A} \wedge \mathcal{A})$$

We inspect each part of this variation. So, by the linearity the exterior derivative d, we have $\delta(d\mathcal{A}) = d(\delta\mathcal{A})$. And since $\delta\mathcal{A} = \delta\mathcal{A}_k dx^k$, we compute:

$$d(\delta\mathcal{A}) = d(\delta\mathcal{A}_k dx^k) = \partial_j(\delta\mathcal{A}_k) dx^j \wedge dx^k,$$

recall that $dx^j \wedge dx^k$ is anti-symmetric, $dx^j \wedge dx^k = -dx^j \wedge dx^k$ is obvious. Then, often times, we write

$$d(\delta\mathcal{A}) = \frac{1}{2}[\partial_j(\delta\mathcal{A}_k) - \partial_k(\delta\mathcal{A}_j)] dx^j \wedge dx^k$$

Now, for the variation of $\mathcal{A}\wedge\mathcal{A}$, apply the Leibniz' rule: $\delta(\mathcal{A}\wedge\mathcal{A}) = \delta\mathcal{A}\wedge\mathcal{A} + \mathcal{A}\wedge\delta\mathcal{A}$, then

$$\begin{aligned}\delta(\mathcal{A}\wedge\mathcal{A}) &= \delta\mathcal{A}_k\mathcal{A}_j dx^k \wedge dx^j + \mathcal{A}_j\delta\mathcal{A}_k dx^j \wedge dx^k \\ &= (\delta\mathcal{A}_j\mathcal{A}_k) - (\mathcal{A}_j\delta\mathcal{A}_k) dx^j \wedge dx^k \text{ by the anti-symmetry}\end{aligned}$$

Combine all to get

$$\begin{aligned}\delta\mathcal{F} &= \delta(d\mathcal{A}) + \delta(\mathcal{A}\wedge\mathcal{A}) \\ &= \partial_j(\delta\mathcal{A}_k) dx^j \wedge dx^k + (\delta\mathcal{A}_j\mathcal{A}_k - \mathcal{A}_j\delta\mathcal{A}_k) dx^j \wedge dx^k\end{aligned}$$

By the definition of covariant derivative ∇, we have the 1-form $\alpha = \alpha^k dx^k$ valued in $End(\mathcal{E})$. This implies that $\nabla_j(\alpha_k) = \partial_j(\alpha_k) + [\mathcal{A}_j, \alpha_k]$. Clearly, $\partial_j(\alpha_k) = \nabla_j(\alpha_k) - [\mathcal{A}_j, \alpha_k]$. Then we compute:

$$\begin{aligned}\partial_j(\delta\mathcal{A}_k) - \partial_k(\delta\mathcal{A}_j) &= [\nabla_j(\delta\mathcal{A}_k) - [\mathcal{A}_j, \delta\mathcal{A}_k]] - [\nabla_k(\delta\mathcal{A}_j) - [\mathcal{A}_k, \delta\mathcal{A}_j]] \\ &= [\nabla_j(\delta\mathcal{A}_k) - \nabla_k(\delta\mathcal{A}_j)] - ([\mathcal{A}_j, \delta\mathcal{A}_k] - [\mathcal{A}_k, \delta\mathcal{A}_j])\end{aligned}$$

,and write covariant expression by explicit anti-symmetrization of the indices (j,k), we get

$$\delta\mathcal{F} = \frac{1}{2}\sum_{j,k}(\nabla_j(\delta\mathcal{A}_j) - \nabla_k(\delta\mathcal{A}_j)) dx^j \wedge dx^k$$

Or equivalently, since we already know that

$$\begin{aligned}d_\nabla(\delta\mathcal{A}) = d_\nabla(\delta\mathcal{A})_k dx^k &= \nabla_j(\delta\mathcal{A})_k dx^j \wedge dx^k \\ &= \frac{1}{2}\sum_{j,k}(\nabla_j(\delta\mathcal{A}_k) - \nabla_k(\delta\mathcal{A}_k)) dx^j \wedge dx^k\end{aligned}$$

we can express the variation form in the sense of exterior differential of the 1-form:

$$\delta\mathcal{F} = d_\nabla(\delta\mathcal{A}) \tag{4.2}$$

In the beginning, we take $\nabla_j(\delta\mathcal{A}_k)$ with resepct to the connection of $End(\mathcal{E})$. But we can also view this by using the Levi-Civita connection on $\Lambda^1(\mathcal{X})\otimes End(\mathcal{E})$, which has torision free: $\Gamma^m_{jk} - \Gamma^m_{kj} = 0$. In particular, consider a 1-form $\mathcal{A}_m dx^m$, we have

$$\nabla_j(\mathcal{A}_k) = \delta(\mathcal{A}_k) - \Gamma^m_{jk}\mathcal{A}_m$$

, and anti-symmetrizing in (j,k) term does not produce extra torision terms. Then, we get the variational form on curvature via components

$$\delta\mathcal{F}_{kj} = \nabla_j(\delta\mathcal{A}_k) - \nabla_k(\delta\mathcal{A}_j) \tag{4.3}$$

Since we have shown the variations on the curvature, we are now ready to compute the variation of the *Yang-Mills functional.*

Theorem 4.2.1 (The Yang-Mills equation)**.** *Suppose that the connection $\mathcal{A}$ is defined by unitary connection and $\delta\mathcal{F}$ be the curvature variation of the 2-form $\mathcal{F}_{\mathfrak{pq}}$. Then the critical points of the Yang-Mills functional $\mathcal{I}(\nabla)$ is*

$$\nabla^{\mathfrak{p}}\mathcal{F}_{\mathfrak{pq}}{}_{\beta}^{\alpha}=0 \tag{4.4}$$

Proof. We claim that the covariant divergence of the curvature is equal to zero. In the process, we first compute the variation of L^2 norm,

$$\begin{aligned}\delta(\|\mathcal{F}\|^2) &= \delta\mathcal{F}\cdot\bar{\mathcal{F}}+\mathcal{F}\cdot\delta\bar{\mathcal{F}}\\ &= \delta\mathcal{F}_{jk}{}_{\beta}^{\alpha}\overline{\mathcal{F}_{\mathfrak{pq}}{}_{\xi}^{\gamma}}H_{\bar{\gamma}\alpha}H^{\beta\bar{\xi}}g^{j\mathfrak{p}}g^{k\mathfrak{q}}+(c.c) \text{ by Leibniz' rule}\\ &= 2(\nabla_j\delta\mathcal{A}_k{}_{\beta}^{\alpha})\overline{\mathcal{F}_{\mathfrak{pq}}{}_{\xi}^{\gamma}}H_{\bar{\gamma}\alpha}H^{\beta\bar{\xi}}g^{j\mathfrak{p}}g^{k\mathfrak{q}}+(c.c) \text{ by curvature variation in components}\end{aligned}$$

Note that $c.c$ is the complex conjugate of $\mathcal{F}$; $j,k,\mathfrak{p},\mathfrak{q}$ be the indices of the base; α,β,γ,ξ be the indices of the fiber. Explicitly, under a variation $\delta\mathcal{A}$ of the connection, we have the curvature variation

$$\delta\mathcal{F}_{jk}{}_{\beta}^{\alpha}=\nabla_j(\delta\mathcal{A}_j{}_{\beta}^{\alpha})-\nabla_k(\delta\mathcal{A}_j{}_{\beta}^{\alpha})$$

Substitute this into the functional, then our variation for *the Yang-Mills functional* becomes

$$\delta\mathcal{I}(\nabla)=2\int_{\mathcal{X}}(\nabla_j\delta\mathcal{A}_k{}_{\beta}^{\alpha})\overline{\mathcal{F}_{\mathfrak{pq}}{}_{\xi}^{\gamma}}H_{\bar{\gamma}\alpha}H^{\beta\bar{\xi}}g^{j\mathfrak{p}}g^{k\mathfrak{q}}\sqrt{g}dx+\int_{\mathcal{X}}(c.c)dx$$

By covariant integration by parts, assuming $\delta\mathcal{A}=0$ on boundary $\partial\mathcal{X}$, we see that the covariant derivative ∇_j of the Hermitian metric H and the base metric g are constant. Then we observe that the covariant derivative form lands on the curvature terms, which is

$$\int_{\mathcal{X}}\nabla_j\delta\mathcal{A}_k{}_{\beta}^{\alpha}\overline{\mathcal{F}_{\mathfrak{pq}}{}_{\xi}^{\gamma}}=-\int_{\mathcal{X}}\delta\mathcal{A}_k{}_{\beta}^{\alpha}\nabla_j(\overline{\mathcal{F}_{\mathfrak{pq}}{}_{\xi}^{\gamma}})$$

Finally, extract the Euler-Lagrange condition

$$\delta\mathcal{I}(\nabla)=0 \text{ for all } \delta\mathcal{A}\,(\text{variation of the connection})$$

this implies

$$\nabla_j(\overline{\mathcal{F}_{\mathfrak{pq}}{}_{\xi}^{\gamma}})=0 \text{ and deduce to get } \nabla^j\mathcal{F}_{j\mathfrak{q}}{}_{\beta}^{\alpha}=0$$

Notice that we have to raise an index via $g^{j\mathfrak{p}}$, we arrive to the equation

$$\nabla^{\mathfrak{p}}\mathcal{F}_{\mathfrak{pq}}{}_{\beta}^{\alpha}=0 \text{ as desired.}$$

Q.E.D.

However, we have restricted the condition of connection $\mathcal{A}$ to be unitary. Strictly speaking, if such arbitrary unitary connection is zero, then it is unclear to get the desired equation. So, for a rigorous argument, we will introduce the orthonormal frames in order to locally define the *Yang-Mills equation.*

4.3 Rigorous Derivation

We begin by showing the definition of orthonormal frame then develop toward the variation of connection and curvature. Briefly speaking, the derivation of curvautre by the use of an orthonormal frame implies uniqueness and existence of *the Yang-Mills functional*.

Definition 4.3.1. Let $e^a = \{e^{\alpha}{}_a\}$ be orthonormal frame, equipped with the fixed Hermitian metric $H_{\bar{\alpha}\beta}$ on complex vector bundle $\mathcal{E} \to \mathcal{X}$. Then, we have 1)$\mathcal{E}_x \cong \mathbb{C}^r$, for a rank r and $x \in \mathcal{X}$ and 2)$H_{\bar{\alpha}\beta}$ on $\mathcal{E}$ is smoothly varying $\langle \cdot, \cdot \rangle_x$ on each fiber $\mathcal{E}_x$. So, having the orthonormal condition means, for each $x \in U$, where $U \in \mathcal{X}$ is open set, we can express the Hermitian metric in simplest form with Kronecker delta function. Concretely, this is

$$H_{\bar{\alpha}\beta} e^{\alpha}{}_a \overline{e^{\beta}{}_b} = \delta_{ab}$$

Now, the following properties are significant result in order to show the variation of the *Yang-Mills functional*.

Proposition 4.3.2. Suppose that the connection form $\mathcal{A}$ is defined by $\nabla e_a = e_c \mathcal{A}^c{}_a$. Then the unitary of ∇ implies the connection $\mathcal{A}$ is unitary.

Proof. Explicitly, we can define the connection by the covariant derivative $\nabla_j e_a = e_c \mathcal{A}_j{}^c{}_a$. i.e. We want to check the connection, such that how the covariant derivative acts on the frame, and hence it suffices to what are the coefficients of the section. First, note that $\mathcal{A}_j{}^c{}_a$ is 1-form valued in the space of endomorphisms. Now, we alredy know that $\langle e_a, e_b \rangle = \delta_{ab}$ (Kronecker delta). Clearly, this is constant; so, we see the derivative $\partial_j \langle e_a, e_b \rangle = 0$. Then, by the definition of the unitarity, we have

$$\langle \nabla_j e_a, e_b \rangle + \langle e_a, \nabla_j e_b \rangle = 0$$

Since we already know the explicit connectino form $\mathcal{A}$, our expression is equal to

$$\langle e_c \mathcal{A}_j{}^c{}_a, e_b \rangle + \langle e_a, e_c \mathcal{A}_j{}^c{}_b \rangle = 0$$

Factor out by the conjugates of the connection, and remember that we're dealing with the complex vector bundle, then

$$\mathcal{A}_j{}^c{}_a \langle e_c, e_a \rangle + \overline{\mathcal{A}_j{}^c{}_b} \langle e_a, e_c \rangle = 0$$

Notice that the inner products are Kronecker delta: $\langle e_c, e_b \rangle = \delta_{cb}$ and $\langle e_a, e_c \rangle = \delta_{ac}$, it follows to get $c = b$ and $a = c$. Hence, we find

$$\mathcal{A}_j{}^b{}_a = \overline{\mathcal{A}_j{}^a{}_b}$$

and thus, we have shown that if the connection $\mathcal{A}$ is unitary, then we must have the relation

$$\mathcal{A}^{\dagger} = -\mathcal{A}$$

This is because, $\mathcal{A}$ as a matrix form, then we can take the interchangable adjoint of row/column indices with Hermitian conjugates. Indeed, for a simple characterization, the matrix form $\mathcal{A}$ is skew-symmetric by the unitarity of the connection. Q.E.D.

Originally, we have seen that the curvature of 2-form is valued in the space of endomorphisms. Now, we can view $End(\mathcal{E})$ as a space of matrices and check the following property.

Proposition 4.3.3. Suppose now, the curvature $\mathcal{F}$ has a unitary connection, equipped with orthonormal frames. Then $\mathcal{F}$ has to be valued via skew-symmetric matrices.

Proof. The analogue is simple. We first take the adjoint of $\mathcal{F}$ and then define through the definition of the curvature in terms of components. We want to show the unitary connection implies the relation: $\mathcal{F}^\dagger = -\mathcal{F}$, with respect to $\mathcal{A}$.

$$\begin{aligned}\mathcal{F}^\dagger &= (d\mathcal{A}+\mathcal{A}\wedge\mathcal{A})^\dagger = \{d(\mathcal{A}_j dx^j) + \mathcal{A}_j dx^j \wedge \mathcal{A}_k dx^k\}^\dagger \\ &= d({\mathcal{A}_j}^\dagger dx^k) + (\mathcal{A}_j\mathcal{A}_k)^\dagger dx^j \wedge dx^k \\ &= d({\mathcal{A}_j}^\dagger dx^j) + {\mathcal{A}_k}^\dagger {\mathcal{A}_j}^\dagger dx^j \wedge dx^k \text{ by anti-symmetry of } \mathcal{A}\text{ ;so, } {\mathcal{A}_j}^\dagger = -A_j \text{ and } {\mathcal{A}_k}^\dagger = -\mathcal{A}_k \\ &= -d\mathcal{A} - \mathcal{A}_j\mathcal{A}_k dx^k \wedge dx^j \text{ by renaming indices } j \leftrightarrow k \\ &= -\mathcal{F}\end{aligned}$$

Thus, we have shown that the curvature $\mathcal{F}$ is a 2-form valued in skew-symmetric matrices. Q.E.D.

Observation 4.3.4. While working with orthonormal frames, we observe that the unitary conncetion of $\mathcal{A}$ and the curvature $\mathcal{F}$ is valued in skew-symmetric matrices. In this sense, instead of keeping the computation by inner products $\langle\cdot,\cdot\rangle$, we take the advanatge of using traces. Let, M,N be skew-symmetric matrices. i.e. $M^\dagger = -M$ or equivalently, $\overline{{M^\alpha}_\beta} = -{M^\beta}_\alpha$. Then,

$$\langle N,M\rangle \equiv {N^\alpha}_\beta \overline{{M^\alpha}_\beta} = -{N^\alpha}_\beta {M^\beta}_\alpha = -Tr(NM) = -Tr(MN)$$

Definition 4.3.5 (The Hodge $*$ operator)**.** Let, φ,ψ be sections of complex vector bundle $\mathcal{E}\to\mathcal{X}$. Then the operator $*$ sends a $\mathfrak{p}-form$ to $(n-\mathfrak{p})-form$ (here $n = dim\mathcal{X}$), such that, $\wedge *$ recovers the set of inner product and proportional to the volume. Which is

$$\varphi \wedge *\bar{\psi} = \langle\varphi,\psi\rangle\sqrt{g}dx, \text{ for any } \varphi,\psi \in \Gamma(\mathcal{X},\Lambda^{\mathfrak{p}})$$

Next, by using the proofs of two propositions above and applying the definition of the Hodge $*$ operator, we can rephrase the *Yang-Mills functional* and directly show the variation. From there, we can derive the *Yang-Mills equation*.

Theorem 4.3.6 (The Yang-Mills functional and its variation)**.** *Initially, we have seen the Yang-Mills functional as*

$$\mathcal{I}(\nabla) = \int_{\mathcal{X}} \|\mathcal{F}\|^2 \sqrt{g}dx$$

Then we rewrite this functional to

$$\mathcal{I}(\nabla) = -\int_{\mathcal{X}} Tr(\mathcal{F}\wedge *\mathcal{F}) \tag{4.5}$$

This implies that the varition of $\mathcal{I}$ becomes

$$\delta\mathcal{I}(\nabla) = 2\langle\delta\mathcal{A}, d_\nabla^\dagger\mathcal{F}\rangle \tag{4.6}$$

Proof. In basic interpretation, we already know that the curvature $\mathcal{F}$ is 2-form valued in $End(\mathcal{E})$. Then, by the definition 4.3.5, we see that

$$\|\mathcal{F}\|^2\sqrt{g}dx = \langle\mathcal{F},\mathcal{F}\rangle\sqrt{g}dx = \mathcal{F}\wedge *\mathcal{F}$$

Since $\mathcal{F}$ is skew symmetric, and by the observation 4.3.4, which we have seen already,

$$\langle\mathcal{F},\mathcal{F}\rangle = Tr(\mathcal{F}^\dagger\mathcal{F}) = Tr((-\mathcal{F})\mathcal{F}) = -Tr(\mathcal{F}^2)$$

Hence, this allows to reformulate the functional as

$$\int_{\mathcal{X}} \|\mathcal{F}\|^2 \sqrt{g} dx = \int_{\mathcal{X}} \langle \mathcal{F}, \mathcal{F} \rangle \sqrt{g} dx = -\int_{\mathcal{X}} Tr(\mathcal{F}^2) \sqrt{g} dx = -\int_{\mathcal{X}} Tr(\mathcal{F} \wedge *\mathcal{F})$$

Next, we compute the variation formula.

$$\begin{aligned}
\delta\mathcal{I}(\nabla) &= -2\int_{\mathcal{X}} Tr(\delta\mathcal{F} \wedge *\mathcal{F} + \mathcal{F} \wedge *\delta\mathcal{F}) \text{ by Leibniz' rule} \\
&= -2\int_{\mathcal{X}} Tr(\delta\mathcal{F} \wedge *\mathcal{F}) \text{ by symmetry of wedge and commutativity of trace} \\
&= -2\int_{\mathcal{X}} Tr(d_\nabla(\delta\mathcal{A}) \wedge *\mathcal{F}) \text{ by equation (4.2)}
\end{aligned}$$

Here, notice that both $\delta\mathcal{A}$ and $\mathcal{F}$ are both skew-symmetric, we find

$$\langle d_\nabla(\delta\mathcal{A}), \mathcal{F} \rangle = \langle \delta\mathcal{A}, d_\nabla^\dagger \mathcal{F} \rangle = -\int_{\mathcal{X}} Tr(d_\nabla(\delta\mathcal{A}) \wedge *\mathcal{F})$$

Thus, we have shown the variation of the *Yang-Mill functional*

$$\delta\mathcal{I}(\nabla) = 2\langle \delta\mathcal{A}, d_\nabla^\dagger \mathcal{F} \rangle$$

Q.E.D.

Proposition 4.3.7 (Explicit forms of the covariant differential $d_\nabla^\dagger$)**.** Let M, N be 1-form and 2-form valued in skew-symmetric matrices, respectively. Then,

1. (The Hodge $*$ operator) $d_\nabla^\dagger = (-1)^{n-1} * d_\nabla *$
2. (Express in terms of components) $(d_\nabla^\dagger N)_{\mathfrak{q}} = -\nabla^j N_{j\mathfrak{q}}$

Proof. We view this through traces and observe first that d_∇ is 1-form.

$$\begin{aligned}
Tr(d_\nabla M \wedge *N) &= Tr(d_\nabla(M \wedge *N)) + Tr(M \wedge d_\nabla * N) \\
&= d\{Tr(M \wedge *N)\} + Tr(M \wedge d_\nabla N)
\end{aligned}$$

Here, we integrate by parts using Stokes' theorem. Which means, in a differential form, if we have φ be $\mathfrak{p}-1$-form and ψ be $\mathfrak{p}$-form, then $\int_X \varphi \wedge *(d\psi) = \int_X \langle \varphi, d\psi \rangle \sqrt{g} dx$. Recall the Stokes' theorem, $\int_X \langle \varphi, d\psi \rangle \sqrt{g} dx = -\int_X \langle \varphi, d^\dagger \psi \rangle \sqrt{g} dx$, where $d^\dagger$ is the formal adjoint of exterior derivative d. Then, as going back to the computation, we find $\int_{\mathcal{X}} d\{Tr(M \wedge *N)\} = 0$ since the boundary term vanishes. This gives the expression

$$\langle d_\nabla M, N \rangle = -\int_{\mathcal{X}} Tr(d_\nabla M \wedge *N) = -\int_{\mathcal{X}} Tr(M \wedge d_\nabla * N)$$

So, we are only left to identify the adjoint $d_\nabla^\dagger$. Now, from a basic Hodge theory, if we have $\mathfrak{p}$-form section φ on $n = dim(\mathcal{X})$, then the operator $*$ satisfies $**\varphi = (-1)^{\mathfrak{p}(n-\mathfrak{p})}\varphi$. So, in our case, where M is 1-form, $** = (-1)^{n-1}$ is clear. Then,

$$-\int_{\mathcal{X}} Tr(M \wedge d_\nabla * N) = -(-1)^{n-1} \int_X M \wedge *(*d_\nabla *)N$$

Observe that $\wedge *$ is the inner product as we have seen already, we get

$$-\int_{\mathcal{X}} Tr(M \wedge d_\nabla * N) = (-1)^{n-1} \langle M, *d_\nabla * N \rangle$$

Hence, we have shown the first part of the proposition. Next, for 2, we express $d_\nabla^\dagger$ by components. From our basic set up, where M is 1-form and N is 2-form, we take

$$d_\nabla M = \sum_{j,k} (\nabla_j M_k) dx^j \wedge dx^k \text{ and } N = \frac{1}{2} \sum_{j,k} N_{kj} dx^j \wedge dx^k$$

Knowing that $(d_\nabla M)_{jk} = \nabla_j M_k - \nabla_k M_j$ and $N_{\mathfrak{pq}}$ is skew-symmetric, we contract the indices:

$$Tr((d_\nabla M)_{jk} N_{\mathfrak{pq}}) g^{j\mathfrak{p}} g^{k\mathfrak{q}} = Tr(\nabla_k (M_k N^{jk})) - Tr(M_j \nabla_k N^{jk}) = -Tr(\nabla_k M_j) N^{jk}$$

where, $N^{jk} = g^{j\mathfrak{p}} g^{k\mathfrak{q}} N_{\mathfrak{pq}}$. The inner product is now

$$\begin{aligned}\langle d_\nabla M, N\rangle &= -\int_{\mathcal{X}} Tr(d_\nabla M)_{jk} N_{\mathfrak{pq}} g^{j\mathfrak{p}} g^{k\mathfrak{q}} \sqrt{g} dx = -\int_{\mathcal{X}} Tr(\nabla_k M_j) N^{jk} \sqrt{g} dx \\ &= -\int_{\mathcal{X}} \nabla_k Tr(M_j N^{jk}) - Tr(M_j \nabla_k N^{jk}) \sqrt{g} dx\end{aligned}$$

Then, integrating by divergence theorem, in particular by the lemma 4.3.8, it is clear to see that $Tr(M_j N^{jk})$ of $\int_{\mathcal{X}} \nabla_k Tr(M_j N^{jk})$ is a vector form. Hence, the whole term vanishes, and thus, the inner product formula reduces to $\langle d_\nabla M, N\rangle = \int_{\mathcal{X}} Tr(M_j \nabla_k N^{jk}) \sqrt{g} dx$. In fact, this implies that the adjoint of the exterior differential is now

$$(d_\nabla^\dagger N)_{\mathfrak{q}} = -\nabla^j N_{j\mathfrak{q}}$$

Q.E.D.

Lemma 4.3.8. *For any vector field v^j, we have*

$$\int_X (\nabla_j v^j) \sqrt{g} dx = 0$$

Proof. By the definition of Levi-Civita connection, the divergence of the vector field is

$$\nabla_j v^j = \partial_j v^j + \Gamma^j_{jk} v^k$$

Indeed, the property of Levi-Civita connection tells us $\nabla_k g_{ij} = 0$, and this yields to $\nabla_j \sqrt{g} = 0$. Then, by the divergence theorem(or equivalently, Stokes' theorem)

$$\int_X (\nabla_j v^j) \sqrt{g} dx = \int_X \partial_j v^j \sqrt{g} dx = \int_{\partial X} \sqrt{g} v^j n_j ds = 0$$

Since the boundary vanishes, we have shown that $\int_X (\nabla_j v^j) \sqrt{g} dx = 0$ Q.E.D.

Theorem 4.3.9 (The Yang-Mills equation)**.** *For a variation $\delta\mathcal{I}(\nabla)$ and the unitary of $\mathcal{A}$, the critical points of the functional $\mathcal{I}(\nabla)$ is the Yang-Mills equation*

$$d_\nabla^\dagger \mathcal{F} = 0 \tag{4.7}$$

Proof. Notice first that since both $\delta\mathcal{A}$ and $\mathcal{A}$ are unitary, we have $\delta\mathcal{I}(\nabla) = 0$ for all $\delta\mathcal{A}$. By proposition 4.3.2, it is clear that both $\delta\mathcal{A}$ and $\mathcal{A}$ are skew-symmetric. However, from the equation(4.6), what we want is $\langle \delta\mathcal{A}, d_\nabla^\dagger \mathcal{F}\rangle = 0$ for all $\delta\mathcal{A}$ and $\mathcal{F}$ be both skew-symmetric. In particular, by proposition 4.3.3, we know that $\mathcal{F}$ is skew-symmeric and so as to $d_\nabla^\dagger \mathcal{F}$. Finally, since $d_\nabla^\dagger \mathcal{F}$ is orthogonal to all variation $\delta\mathcal{A}$, we conclude to get $d_\nabla^\dagger \mathcal{F} = 0$ as desired. Another way to prove the *Yang-Mills equation* is followed by the second part of proposition 4.3.7. From $(d_\nabla^\dagger N)_{\mathfrak{q}} = -\nabla^j N_{j\mathfrak{q}}$ where N is a 2-form, we apply this to the curvature $\mathcal{F}_{j\mathfrak{q}}{}^\alpha_\beta$, which is also a 2-form. Then, the divergence of the curvature: $\nabla^j \mathcal{F}_{j\mathfrak{q}}{}^\alpha_\beta = d_\nabla^\dagger \mathcal{F} = 0$ is the *Yang-Mills equation.* Q.E.D.

Chapter 5: Examples and Solutions of The Yang-Mills Equation

In this chapter, we inspect the examples and solutions of the *Yang-Mills equation.*

5.1 The Maxwell's Equation

From our rigorous approach in previous section, in particular, followed by the result of the proposition 4.3.7 and theorem 4.3.9 , $(d_{\nabla}^{\dagger}N)_{\mathfrak{q}} = -\nabla^j N_{j\mathfrak{q}}$ is absolutely powerful and elegant way of recapturing the base case of the equations of the electromagnetism. In general physics, such equations that describe the interaction between electricity and magnetic fields, arising from their charges and currents, are the famous *Maxwell's equation.* Precisely, the classical Maxwell's equation consists of four distinct different partial differential equations, and these are the Gauss's law of electricity, the Gauss's law of magnetism, the Maxwell-Faraday's law of induction, and the Maxwell-Ampére's law.

Definition 5.1.1. Let E be electric fields, B be magnetic fields and ρ,σ represent the charge and current densitites, respectively. Then we state the Maxwell's equation:

$$\nabla \cdot \vec{E} = \rho \tag{5.1}$$

$$\nabla \cdot \vec{B} = 0 \tag{5.2}$$

$$\nabla \times \vec{E} + \partial_t \vec{B} = 0 \tag{5.3}$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = \sigma \tag{5.4}$$

In the process of illustrating the formalism, we apply $U(1)$ *Yang-Mills* theory on Minkowski spacetime $\mathbb{R}^{1,3}$ of 4-dimensional Riemmanian manifolds. This means, we view $t = x^0$ as the time coordinate and $\vec{x} = x^j$ for $1 \leq j \leq 3$ as the spatial coordinates, where the Lorentz metric is given by

$$ds^2 = dt^2 - \sum_{j=1}^{3} \left(dx^j\right)^2$$

Remark 5.1.2. $U(1)$ is the simplest form of a Yang-Mills field. It is abelian, compact, and connected; significantly, it associates with a vector bundle whose fibers are 1-dimensional complex vector spaces, which is clearly a complex line bundle (cf. §2.1). So, it provides a geometrically perfect setting for describing electromagnetism.

Definition 5.1.3. The connection $\mathcal{A}$ is a 1-form. We can express it as in a vector form

$$\mathcal{A} = \sum_{j=0}^{3} \mathcal{A}_j dx^j = -\varphi dt + \sum_{j=1}^{3} \mathcal{A}_j dx^j$$

Here, $-\varphi dt$ is the scalar potential and the remaining sum is the vector potential part of $\mathcal{A}$.

Definition 5.1.4. Since our connection $\mathcal{A}$ is a scalar (i.e. abelian), it is obvious to see $\mathcal{A} \wedge \mathcal{A} = 0$. Then, we define the curvature $\mathcal{F}$ as follows

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} + d\mathcal{A} \wedge \mathcal{A} = d\mathcal{A} \\ &= \frac{1}{2} \sum_{j,k=0}^{3} (\partial_j \mathcal{A}_k - \partial_k \mathcal{A}_j) dx^j \wedge dx^k = \frac{1}{2} \mathcal{F}_{kj} dx^j \wedge dx^k \end{aligned}$$

In fact, in physics literature, such definition of the curvature 2-form

$$\mathcal{F}_{kj} = \frac{1}{2}\left(\frac{\partial \mathcal{A}_k}{\partial x^j} - \frac{\partial \mathcal{A}_j}{\partial x^k}\right)$$

is referred to as the field strength of the $U(1)$ Yang-Mills field (i.e. Gauge field). Note that since $\mathcal{F}_{jk}$ is antisymmetric, there is no curvature component for $\mathcal{F}_{(0,0)}$ as $dx^0 \wedge dx^0 = 0$.

Definition 5.1.5. Using the definition of the curvature 2-form $\mathcal{F}$, we can show the electric and magnetic field. The *electric field* $\vec{E} = (E_1, E_2, E_3)$ is given by time-space components of the curvature

$$E_j = \mathcal{F}_{j0} \text{ for } j = 1,2,3$$

If $j \neq k$, then the *magnetic field* $\vec{B} = (B_1, B_2, B_3)$ is defined through space components. So, for each $j = 1,2,3$, let

$$B_j = \mathcal{F}_{kl}\text{ , where } (j,k,l) \text{ is a cyclic permutation of } (1,2,3)$$

We can represent this in 4×4 matrix of the 2-form curvature $\mathcal{F}_{kj}$, where the first row and column be the *electric field* and the rest of antisymmetric terms be the *magnetic field*, which means,

$$F_{kj} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \text{since } E_j = \mathcal{F}_{j0}, -E_j = \mathcal{F}_{0j} \text{ and } \mathcal{F}_{12} = B_3, \mathcal{F}_{23} = B_1, \mathcal{F}_{13} = B_2$$

Now, we have recognized the fact that these physical notions are purely components of the curvature. Then, we are now ready to show the Maxwell's equation with respect to the framework of the *Yang-Mills equation*.

Theorem 5.1.6. *The Maxwell's equation provides a characterization of the electromagnetism. In particular, from the $U(1)$ Yang-Mills theory, the homogenous Maxwell's equation can be derived from the Second Bianchi Identity and the inhomogenous Maxwell's equation follwed by the adjoint of the exterior differential of the curvature.*

Proof. We show this by considering the simplest condition: the Maxwell's equation in a vacuum state, where the charge and current densities, ρ and σ, vanish from the equations (5.1) and (5.4). Recall from the definition 3.1.1. and since $U(1)$ is abelian, the Second Bianchi Identity is $d\mathcal{F} = 0$. Geometrically, the homogenous equation has no charges and currents, and since the curvautre $\mathcal{F}$ is a closed 2-form, it naturally arises from the Second Bianchi Identity. Applying this into the Maxwell's equation, which is equivalent to say in componentwise

$$\partial_l \mathcal{F}_{kj} + \partial_k \mathcal{F}_{jl} + \partial_j \mathcal{F}_{lk} = 0$$

For the one case, if none of the indices, j,k,l is zero (i.e. all the indices are space components), then we can rewrite this as the divergence of the magnetic field by the antisymmetry of $\mathcal{F}kj$. Hence, we have shown the equation (5.2)

$$\nabla \cdot \vec{B} = 0$$

The other case is that whenever one of the indices j,k,l is zero. Without loss of generality, if we set $l = 0$, then the remaining indices, j,k, must be defined via space components. For instance, we can take the case when $k = 1$ and $j = 2$ for $l = 0$. Then the Second Bianchi Identity becomes

$$\partial_t \mathcal{F}_{12} + \partial_1 \mathcal{F}_{20} + \partial_2 \mathcal{F}_{01} = 0$$

Followed by the definition 5.1.5, which is now

$$\partial_t B_3 + \partial_1 E_2 - \partial_2 E_1 = 0$$

Putthing altogether, and repeating this procedure by the cyclic permutation of indinces, we can rewrite this as the curl of the electric field; so, we have shown the equation (5.3)

$$\partial_t \vec{B} + \nabla \times \vec{E} = 0$$

For the remaining two equations, it is followed by the theorem 4.3.9 and the unitarity of the connection $\mathcal{A}$. Then we observe the *Yang-Mills equation* takes the form

$$d_{\nabla}^{\dagger}\mathcal{F} = 0 \Longleftrightarrow \partial_0 \mathcal{F}_{0k} - \sum_{j=1}^{3} \partial_j \mathcal{F}_{jk} = 0$$

, where the Minkowski metric(i.e. base metric) is diagonal. Which means, $g^{j\mathfrak{q}} = g_{j\mathfrak{q}} = 1$ for $j = \mathfrak{q} = 0$ and -1 for $1 \leq j = \mathfrak{q} \leq 3$. Then we consider two cases. For $k = 0$, then it is clear to see $\partial_0 \mathcal{F}_{0k} = 0$; consequently,

$$-\sum_{j=1}^{3} \partial_j \mathcal{F}_{jk} = -\nabla \cdot \vec{E} = 0$$

Hence, the above equation is now the divergence of the electric field; thus, the Gauss's law of electricity (5.1) in vacuum state becomes

$$\nabla \cdot \vec{E} = 0 \tag{5.5}$$

Next, for $k \neq 0$, which refers that $k \in \{1,2,3\}$, and again, by the antisymmetry,

$$\mathcal{F}_{k0} = E_k \text{ and } \mathcal{F}_{0k} = -E_k; \text{ so, } \partial_0 \mathcal{F}_{0k} = \partial_0(-E_k) = -\partial_t E_k$$

i.e. the first term becomes the time derivative of electric field. The remaining part, $-\sum_{j=1}^{3} \partial_j \mathcal{F}_{jk}$, transforms to the curl of the magnetic field, and this is easily observed by the 4×4 matrix in the definition 5.1.5. Thus, the Maxwell-Ampére's law (5.4) in vacuum state is now

$$\nabla \times \vec{B} - \partial_t \vec{E} = 0 \tag{5.6}$$

To conclude, we have shown that the $U(1)$ *Yang-Mills theory* generalizes and recovers the *Maxwell's equation.* Q.E.D.

5.2 Self-Dual and Anti Self-Dual Solutions

These solutions are notably famous and simple case in topology. In fact, it is a special condition when our base manifold is a 4-dimensional Riemmanian manifold.

Proposition 5.2.1. Let $\mathcal{X}$ be a 4-dimensional Riemmanian manifold equipped with the metric g. Then the Hodge star operator $*$ is a self adjoint, and consequently, we may decompose the 2-form, denoted $\Lambda^2(\mathcal{X})$, into orthogonal direct sum as follows

$$\Lambda^2(\mathcal{X}) = \Lambda^2_+(\mathcal{X}) \bigoplus \Lambda^2_-(\mathcal{X})$$

, where $\Lambda^2_\pm(\mathcal{X})$ be the eigenspace of $*$ with eigenvalues ± 1.

Proof. Followed by the definition 4.3.5, the Hodge star operator $*$ maps $\mathfrak{p}$-form to $n-\mathfrak{p}$-form. Which means,

$$*:\Lambda^{\mathfrak{p}} \to \Lambda^{n-\mathfrak{p}} \text{ so if } \mathfrak{p}=2, n=4, \text{ then } *:\Lambda^2 \to \Lambda^2$$

Now, again by the definition, we wish show the identity of the Hodge $*$ operator, such that, $\varphi \wedge *\psi = \langle \varphi, \psi \rangle \sqrt{g}dx$, where $\sqrt{g}dx$ is the Riemmanian volume form and φ, ψ be any real-valued 2-form. We compute

$$\langle *\varphi, \psi \rangle \sqrt{g}dx = (*\varphi) \wedge (*\psi) = (*\psi) \wedge (*\varphi) = \langle *\psi, \varphi \rangle \sqrt{g}dx = \langle \varphi, *\psi \rangle \sqrt{g}dx$$

and since $\sqrt{g}dx \neq 0$, we have shown

$$\langle *\varphi, \psi \rangle = \langle \varphi, *\psi \rangle$$

so that $*$ is clearly a self adjoint on $\Lambda^2(\mathcal{X})$. Next, since our case is when $\mathfrak{p}=2$ and $n=4$, we find

$$** = (-1)^{(\mathfrak{p})\cdot(n-\mathfrak{p})} = (-1)^{2\cdot 2} = 1$$

Hence, $*^2 = 1$, and the eigenvalues of $*$ have to be ± 1. So, the self adjoint operator $*$ can be diagonalizable, and its eigenspaces are the self-dual and the anti self-dual of 2-forms, respectively

$$\Lambda^2_+(\mathcal{X}) = \{\varphi \in \Lambda^2(\mathcal{X}) | *\varphi = \varphi\} \text{ and } \Lambda^2_-(\mathcal{X}) = \{\varphi \in \Lambda^2(\mathcal{X}) | *\varphi = -\varphi\}$$

Notice that these two eigenspaces are pointwise orthogonal and span all of $\Lambda^2(\mathcal{X})$; thus, yielding toward the decomposition as

$$\Lambda^2(\mathcal{X}) = \Lambda^2_+(\mathcal{X}) \bigoplus \Lambda^2_-(\mathcal{X})$$

Q.E.D.

Proposition 5.2.2. Let $\mathcal{F}$ be any 2-form curvature of a connection $\mathcal{A}$ on a complex vector bundle $\mathcal{E} \to \mathcal{X}$, where $\mathcal{X}$ is a smooth 4-dimensional Riemmanian manifold. Then, applying the self-dual and anti self-dual decomposition of the curvature, the Second Chern-class is topologically invariant.

Proof. By the same analogue from previous proposition, the 2-form curvature $\mathcal{F}$ decomposes as follows

$$\mathcal{F}^a{}_b = (\mathcal{F}_+)^a{}_b + (\mathcal{F}_-)^a{}_b \text{ ,where } (\mathcal{F}_\pm)^a{}_b \in \Lambda^2_\pm(\mathcal{X}) \otimes End(\mathcal{E})$$

Then, recall from the chapter 3, especially focusing on the Chern-Weil theory, we see the second characteristic class

$$C_2(\mathcal{E}) = Tr(\mathcal{F} \wedge \mathcal{F})$$

Since our connection is independent, we may choose any. Then we compute the decomposition

$$\begin{aligned}
Tr(\mathcal{F} \wedge \mathcal{F}) &= (\mathcal{F}_+ + \mathcal{F}_-)^a{}_b \wedge (\mathcal{F}_+ + \mathcal{F}_-)^b{}_a \\
&= (\mathcal{F}_+)^a{}_b \wedge (\mathcal{F}_+)^b{}_a + (\mathcal{F}_-)^a{}_b \wedge (\mathcal{F}_-)^b{}_a + (\mathcal{F}_+)^a{}_b \wedge (\mathcal{F}_-)^b{}_a + (\mathcal{F}_-)^a{}_b \wedge (\mathcal{F}_+)^b{}_a \\
&\text{and by the Hodge} * \text{ identities, such that, } *\mathcal{F}_+ = \mathcal{F}_+ \text{ and } *\mathcal{F}_- = -\mathcal{F}, \\
&= (\mathcal{F}_+)^a{}_b \wedge *(\mathcal{F}_+)^b{}_a - (\mathcal{F}_-)^a{}_b \wedge *(\mathcal{F}_-)^b{}_a
\end{aligned}$$

It followed from the fact that each $\mathcal{F}_+$ and $\mathcal{F}_-$ has $+1, -1$ eigenvalue, respectively, and their eigenspaces $\Lambda_\pm(\mathcal{X})$ are mutually orthogonal to each other; hence, the cross terms vanish. i.e.$(\mathcal{F}_+)^a{}_b \wedge *(\mathcal{F}_-)^c{}_d = 0$. Also, it is clear to see

$$(\mathcal{F}_+)^a{}_b \wedge *(\mathcal{F}_+)^b{}_a = \|\mathcal{F}_+\|^2 \text{ ,and so as to } (\mathcal{F}_-)^a{}_b \wedge *(\mathcal{F}_-)^b{}_a = \|\mathcal{F}_-\|^2$$

Putting altogether and integrating over $\mathcal{X}$, we obtain the Chern class

$$\int_{\mathcal{X}} C_2(\mathcal{E}) = \int_{\mathcal{X}} Tr(\mathcal{F}\wedge\mathcal{F}) = \int_{\mathcal{X}} (\mathcal{F}_\pm)^a{}_b \wedge *(\mathcal{F}_\pm)^b{}_a$$
$$= \|\mathcal{F}_+\|^2_{L^2} - \|\mathcal{F}_-\|^2_{L^2}$$

This is in fact the second Chern-class which is topologically invariant as it remains unchanged under continuous decomposition. Q.E.D.

Now, we are going back to the *Yang-Mills functional* and deriving the solutions.

Theorem 5.2.3. *From 4-dimensional Riemmanian manifold $\mathcal{X}$, we have a connection $\mathcal{A}$ over complex vector bundle $\mathcal{E}\to\mathcal{X}$, with 2-form curvature decomposed as $\mathcal{F}^a{}_b = (\mathcal{F}_+)^a{}_b + (\mathcal{F}_-)^a{}_b$. Then, the Yang-Mills functional is the lower bound of the second Chern-class and the equality holds if and only if $\mathcal{F}_- = 0$ or $\mathcal{F}_+ = 0$. These are the self-dual and anti self-dual solutions of the functional.*

Proof. The proof immediately follows by the proposition 5.2.2. and the equation(4.1) the definition of the *Yang-Mills functional*. Since we already know the decomposition of the 2-form curvature with $*\mathcal{F}_\pm = \pm\mathcal{F}_\pm$, we can express the functional as follows

$$\mathcal{A} \to \mathcal{I}(\mathcal{A}) = \int_{\mathcal{X}} \|\mathcal{F}_{\mathcal{A}}\|^2 \sqrt{g}dx = \int_X \left(\|\mathcal{F}_+\|^2 + \|\mathcal{F}_-\|^2\right)\sqrt{g}dx$$

We also know that $\|\mathcal{F}_{\mathcal{A}}\|^2$ is L^2 norm of the curvature with connection $\mathcal{A}$ associated with the volume form $\sqrt{g}dx$, so we can conveniently rewrite as

$$\mathcal{I}(\mathcal{A}) = \|\mathcal{F}_+\|^2_{L^2} + \|\mathcal{F}_-\|^2_{L^2}$$

Then, using the result of the proposition above, the second Chern-class tells us the inequality

$$\mathcal{I}(\mathcal{A}) = \|\mathcal{F}_+\|^2_{L^2} + \|\mathcal{F}_-\|^2_{L^2} \geq |\ \|\mathcal{F}_+\|^2_{L^2} - \|\mathcal{F}_-\|^2_{L^2}\ | = |\int_{\mathcal{X}} C_2(\mathcal{E})|$$

Hence, we have shown that the *Yang-Mills functional* is bounded below by the second Chern-class. Observe that whenever the lower bound is saturated, so that either components of the second Chern-class goes to zero. i.e.We must have either $\|\mathcal{F}_+\|^2_{L^2} = 0$ or $\|\mathcal{F}_-\|^2_{L^2}$. Then, we have two cases to consider. If $\mathcal{F}_- = 0$, then $\mathcal{F} = \mathcal{F}_+$ which satisfies $\mathcal{F} = *\mathcal{F}$, the connection is self-dual. If $\mathcal{F}_+ = 0$, then $\mathcal{F} = \mathcal{F}_-$, satisying the connection to be anti self-dual since $\mathcal{F} = -*\mathcal{F}$. Which means, we have found that the minima of the *Yang-Mills functional* given by the connections. Therefore, the critical points of this functional, which are solutions of the *Yang-Mills equation* are defined through the connections of self-dual and anti self-dual. Q.E.D.

5.3 The Belavin-Polyakov-Schwartz-Tyuptin Instantons

In 1975, theoretical physicists, Belavin, Polyakov, Schwartz, and Tyuptin showed a pseudoparticle solutions of the *Yang-Mills equation*; in a simpler term, we call it the *BPST-instantons* [1]. In fact, it is well known as a classical solution to $SU(2)$ *Yang-Mills theory* in euclidean $\mathbb{R}^4$ space.

Definition 5.3.1. The definition of *pseudoparticle solution* is that in $\mathbb{R}^4$ euclidean space, the *Yang-Mills functional* $\mathcal{I}(\mathcal{A})$ minimized locally by the connection(i.e.long-range fields) $\mathcal{A}$. Precisely, recall from the chapter 4 on the Yang-Mills functional, we can express this as follows

$$\mathcal{I}(\mathcal{A}) = \int_{\mathcal{X}=\mathbb{R}^4} \|\mathcal{F}\|^2 \sqrt{g}dx < \infty \text{ which is finite}$$

For a better understanding, we consider $S^3 \subset \mathbb{R}^4$ as the basic case. The *BPST-instantons* tells us that for a connection $\mathcal{A}$, the curvature satisfies the pointwise dacay as reaching near the boundary of S^3. This means, we have $\|\mathcal{F}\| \to 0$ as $|x| \to \infty$;so, for a gauge transformation, we have a smooth map $\mathfrak{g} : {S^3}_\infty \to SU(2)$, defined through asymptotical 1-form connection $\mathcal{A}$, such that,

$$\mathcal{A}_j \approx \mathfrak{g}^-1(x)\frac{\partial \mathfrak{g}(x)}{\partial x_j} \text{ for sufficiently large } x \text{ ,where } \mathfrak{g}(x) \text{ are matric form of } SU(2)$$

We compute the curvature, followed by the usual definition in section(2.3),

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = g^{-1}d(gdg^{-1}) = g^{-1}dg \wedge dg^{-1} = 0$$

In order to find the abolute minumum of the phase in componentwise, we apply characterization, and that is the Second Chern-class in $\mathbb{R}^4$ given by normalization

$$C_2(\mathcal{E}) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} Tr(\mathcal{F} \wedge \mathcal{F})$$

Since $C_2(\mathcal{E}) = k$ is the integer, we claim this to be the instanton number, such that, the topological charge $k = 1$. Now, for a self-duality, we already know that $\mathcal{F} = *\mathcal{F}$. Then by the equation(4.5), we can rewrite the functional as

$$\mathcal{I}(\mathcal{A}) = \int_{\mathcal{X} = \mathbb{R}^4} \|\mathcal{F}\|^2 \sqrt{g} dx = \int_{\mathbb{R}^4} Tr(\mathcal{F} \wedge *\mathcal{F}) = \int_{\mathbb{R}^4} Tr(\mathcal{F} \wedge \mathcal{F})$$

Finally, by our assumtion, we have reached the solution

$$\mathcal{I}(\mathcal{A}) = \int_{\mathbb{R}^4} Tr(\mathcal{F} \wedge \mathcal{F}) = 8\pi^2 C_2(\mathcal{E}) = 8\pi^2$$

Indeed, from the result of previous section, we can see this as $\mathcal{I}(\mathcal{A}) \geq |\int_{\mathcal{X}} C_2(\mathcal{E})|$. Hence, the abolute minumum is when the equality holds for $8\pi^2$, which represents the *BPST-instantons*.

Chapter 6: The Yang-Mills Equation on Kähler Geometry

Previously, we have seen the *Yang-Mills equation* in arbitrary smooth complex vector bundles. This is absolutely complicated based on the results above. We now assume that such smooth bundles are holomorphic so that we can derive the equation in a much simpler case. In fact, the holomorphicity has more structure, and the result would be extremely beautiful. In the process, we consider the condition of a smooth vector bundle, $\mathcal{E} \to \mathcal{X}$ to be a holomorphic vector bundle over a *Kähler* manifold, denoted $(\mathcal{X}, \omega_{\mathcal{X}})$.

6.1 Properties of Kähler Manifolds

The analogue followed by the chapter 2, where we have demonstrated the notions of vector bunldes, connections, and curvature. Since we already know how to treat curvatures and connections on Hermitian holomorphic bundles $\mathcal{E} \to \mathcal{X}$, eqiupped with Hermitian metric $H_{\bar{\alpha}\beta}$, the important case to see in particular is when $\mathcal{X}$ is a complex manifold, and $\mathcal{E} = T^{1,0}$ be the space of (1,0)-form vector fields on $\mathcal{X}$. We begin by showing basic definitions.

Definition 6.1.1. We retrieve from the definition of complex vector bundles, such that $\mathcal{E} = \Lambda^{1,0}$ be the bundle defined through the space of (1,0)-from on $\mathcal{X}$. In a local coordiate system,

$$z_\mu{}^l \text{ for } 1 \leq l \leq dim(\mathcal{X}) = n, \text{ the smooth sections } \varphi = \sum_{l=1}^{n} dz_\mu{}^l \varphi_{l,\mu}.$$

For a different coordiate system, we have

$$dz_\mu{}^l \varphi_{l,\mu} = \varphi = dz_\nu{}^m \varphi_{m,\nu}, \text{ and this implies } \varphi_{l,\mu} = \frac{\partial z_\nu{}^m}{\partial z_\mu{}^l} \varphi_{m,\nu}.$$

Clearly, $\mathcal{E} = \Lambda^{1,0}$ is defined by the holomorphic transition functions: $\frac{\partial z_\nu{}^m}{\partial z_\mu{}^l}$. Now, the holomorphic *tangent bundle*, denoted $T^{1,0}$, is locally the dual bundle to $\mathcal{E} = \Lambda^{1,0}$. This means, $T^{1,0}$ has sections $V = V_\mu{}^l \frac{\partial}{\partial z_\mu{}^l}$, and the transition functions are given by $V_\mu{}^l = \frac{\partial z_\nu{}^m}{\partial z_\mu{}^l} V_\nu{}^m$. Then, we take the vector bundles of the vector fields and obtain a *tangent bundle*

$$\mathcal{E} = T^{(1,0)}\mathcal{X} = \{V = V^l \frac{\partial}{\partial z^l}\}$$

For a notational consistency, we set $\Gamma(\mathcal{X}, T^{1,0}) \ni V = V^l$ be the smooth sections of tangent bundles (formerly, it was φ^α); $g_{\bar{l}m}$ be the Hermitian metric of a tangent bundle; $R_{\bar{k}l}{}^p_q$ be its curvature. In fact, there exists a lot of possibilities that such identities for the curvature of a tangent bundle would not make sense for the curvature of a general holomorphic vector bundle; we must introduce the *Kähler* metrics in turn.

Definition 6.1.2. Given a metric $g_{\bar{l}m}$ of a tangent bundle $T^{1,0}$, we associate the following (1,1)-form by $\omega = i g_{\bar{l}m} dz^m \wedge d\bar{z}^l$. The metric $g_{\bar{l}m}$ is said to be *Kähler*, if $d\omega = 0$, where ω is closed in global condition. Or, explicitly, we can express this in local coordinates

$$\frac{\partial}{\partial z^p} g_{\bar{l}m} = \frac{\partial}{\partial z^m} g_{\bar{l}p} \text{ and } \frac{\partial}{\partial \bar{z}^p} g_{\bar{l}m} = \frac{\partial}{\partial \bar{z}^l} g_{\bar{p}m}$$

Computing via $d\omega = 0$, we can expand it as

$$\frac{\partial g_{\bar{l}m}}{\partial z^p} dz^p \wedge dz^m \wedge d\bar{z}^l + \frac{\partial g_{\bar{l}m}}{\partial \bar{z}^p} d\bar{z}^p \wedge dz^m \wedge d\bar{z}^l = 0$$

Notice the symmetry, we can also view the *Kähler* potential locally for some smooth function ψ,

$$g_{\bar{l}m} = \frac{\partial^2 \psi}{\partial z^m \partial \bar{z}^l}$$

Definition 6.1.3. Putting definitions 6.1.1. and 6.1.2. altogether, a complex manifold $\mathcal{X}$, equipped with a Hermitian metric $g_{\bar{l}m}$ on its tangent bundle $T^{1,0}$, is called a *Kähler* manifold if the associated (1,1)-form (the *Kähler* form)

$$\omega = i g_{\bar{l}m} dz^m \wedge d\bar{z}^l$$

is closed. i.e. $d\omega = 0$.

We can now derive the identities for the curvature of the *Kähler* metrics.

Proposition 6.1.4. Let $\mathcal{X}$ be a *Kähler* manifold having the metric $g_{\bar{p}m}$. We define the curvature tensor locally by lowering the indices as follows:

$$R_{\bar{k}j\bar{p}q} = (R_{\bar{k}j}{}^m{}_q) g_{\bar{p}m}$$

Then, the curvature tensor satisfies the characteristic conditions of symmetry, such that,

$$R_{\bar{k}j\bar{p}q} = R_{\bar{p}j\bar{k}q} = R_{\bar{p}q\bar{k}j}$$

i.e. We can freely permute anti-holomorphic indices $\bar{k}, \bar{p}$ and holomorphic indices j, q, which is only true for *Kähler* metrics.

Proof. Recall the Chern-connection ∇ of vector bundles, we already know

$$\nabla_{\bar{l}} \varphi = \partial_{\bar{l}} \varphi^\alpha \text{ and}$$

$$\nabla_l \varphi^\alpha = H^{\alpha\bar{\beta}} \partial_l (H_{\bar{\beta}\alpha} \varphi^\alpha) = \partial_l \varphi^\alpha + (H^{\alpha\bar{\beta}} \partial_l H_{\bar{\beta}\gamma}) \varphi^\gamma \text{ by Leibniz' rule.}$$

In a metrix notation, where $\mathcal{A}_l = H^{-1} \partial_l H$, so that $\nabla_l = \partial_l + \mathcal{A}_l$, and we see $H^{\alpha\bar{\beta}} \partial_l H_{\bar{\beta}\gamma} = \mathcal{A}^\alpha{}_l \gamma$; so, $\mathcal{A} = \partial_z{}^l (H^{-1} \partial_l H)^\alpha_\gamma$. For a curvature, where $\mathcal{F}_{\bar{m}l} = [\nabla_{\bar{m}}, \nabla_l] = -\partial_{\bar{m}} \mathcal{A}_l$, and locally this is

$$\mathcal{F}_{\bar{m}l}{}^\alpha_\gamma = -\partial_{\bar{m}} \mathcal{A}_l{}^\alpha_\gamma = -\partial_{\bar{m}} (H^{-1} \partial_l H)^\alpha_\gamma$$

Then, applying this convention to the tangent bundles, we find

$$R_{\bar{m}l}{}^p_q = -\partial_{\bar{m}} (g^{p\bar{r}} \partial_l g_{\bar{r}q}) = g^{p\bar{t}} (\partial_{\bar{m}} g_{\bar{t}s}) g^{s\bar{r}} \partial_l g_{\bar{r}q} - g^{p\bar{r}} \partial_{\bar{m}} \partial_l g_{\bar{r}q}$$

Finally, lowering the indices, which is (1,1)-form valued in $End(T^{1,0})$,

$$\begin{aligned} R_{\bar{m}l\bar{u}q} &= g_{\bar{u}p} R_{\bar{m}l}{}^p_q = g_{\bar{u}p} \{ g^{p\bar{t}} (\partial_{\bar{m}} g_{\bar{t}s}) g^{s\bar{r}} \partial_l g_{\bar{r}q} - g^{p\bar{r}} \partial_{\bar{m}} \partial_l g_{\bar{r}q} \} \\ &= \partial_{\bar{m}} g_{\bar{u}s} g^{s\bar{r}} \partial_l g_{\bar{r}q} - \partial_{\bar{m}} \partial_l g_{\bar{u}q} \end{aligned}$$

Since $\partial_{\bar{m}} g_{\bar{u}s} = \partial_{\bar{u}} g_{\bar{m}s}$, we obtain the desired identity: $R_{\bar{m}l\bar{u}q} = -\partial_{\bar{m}} \partial_l g_{\bar{u}q}$. Observe now that the partial derivatives are commutative and $g_{\bar{u}q} = \overline{g_{\bar{q}v}}$, we easily find

$$R_{\bar{m}l\bar{u}q} = R_{\bar{u}l\bar{m}q} \text{ (permute } \bar{m} \leftrightarrow \bar{u}) \text{ and } R_{\bar{m}l\bar{u}q} = R_{\bar{m}q\bar{u}l} \text{ (permute } l \leftrightarrow q)$$

Thus, we have shown the *Kähler* symmetries of the curvature tensor

$$R_{\bar{k}j\bar{p}q} = R_{\bar{p}j\bar{k}q} = R_{\bar{p}q\bar{k}j}$$

Q.E.D.

Now, we can also show the Ricci curvature. It is defined as (1,1)-form, such that, for all ω be *Kähler*,

$$Ric(\omega) = idz^j \wedge d\bar{z}^k R_{\bar{k}j}{}^m{}_m \text{, where } R_{\bar{k}j} \text{ obtained from the trace of the general curvature}$$

The general formual can be easily observable when the metric G is diagonalizable as follows

$$\delta(log\, det(G)) = \delta log(\pi\lambda_j) = \sum_j \delta log\, \lambda_j = \sum \frac{\delta\lambda_j}{\lambda_j} = Tr(G^{-1}\delta G)$$

Then, we see that

$$R_{\bar{k}j} = R_{\bar{k}j}{}^m{}_m = -\partial\bar{k}\left(g^{m\bar{r}}\partial_j g_{\bar{r}m}\right) = -\partial\bar{K}\partial_j(log\, det g_{\bar{p}q})$$

Hence, we find

$$Ric(\omega) = i\partial\bar{\partial}(log\, det g_{\bar{p}q})$$

Notice that $Ric(\omega)$ is closed and we recognize the fact that the first Chern-class is a de Rham cohomology class of Ricci curvature; i.e.we set $C_1(\mathcal{E}) = [Ric(\omega)]_{dR}$. So with $\mathcal{E} = T^{1,0}$, and since $R_{\bar{k}j}{}^p{}_p = Tr\mathcal{F}$ where $\mathcal{F} = R_{\bar{k}j}{}^p{}_q$, we observe $C_1(\mathcal{X}) = iC_1(\mathcal{E})$.

6.2 The Hermitian-Einstein Equation

Understanding the key feature of *Kähler* manifolds, we introduce *Hermitian-Einstein equation.* Though, we won't show fully explicit proof since the stability is not our goal.

Definition 6.2.1. Let $\mathcal{E} \to \mathcal{X}$ be an irreducible holomorphic vector bundle over a compact *Kähler* manifold(definition 6.1.3). Then we fix a reference metric on $\mathcal{E}$, denoted $\hat{H}_{\bar{\alpha}\beta}$. For another given Hermitian metric $H_{\bar{\alpha}\beta}$ on $\mathcal{E}$, we define its corresponding endomorphism, denoted $\mathfrak{H}^\alpha_\beta = \hat{H}^{\alpha\bar{\gamma}} H_{\bar{\gamma}\beta}$. Express this in matrix form, $\mathfrak{H} = \hat{H}^{-1}H$. Then we show the *Hermitian-Einstein equation* as follows

$$g^{j\bar{k}}(\mathcal{F}_\epsilon)_{\bar{k}j}{}^\alpha{}_\beta - \mu\delta^\alpha_\beta = -\epsilon(log\mathfrak{H}_\epsilon)^\alpha_\beta \tag{6.1}$$

Again, in matrix, note that $\mathcal{F}_\epsilon$ is the curvature of $(H_{\bar{\alpha}\beta})_\epsilon$,

$$\Lambda\mathcal{F}_\epsilon - \mu I = \epsilon \log\mathfrak{H}_\epsilon \tag{6.2}$$

From our endomorphism $\mathfrak{H}$ above, we let $\mathfrak{H}_\epsilon = \hat{H}^{-1}H_\epsilon$ be the solution of the *Hermitian-Einstein equation* for any $0 < \epsilon < 1$.

6.3 The Yang-Mills Equation on Compact Kähler Manifolds

Stretching from the last two sections, where we have learned about *Kähler* manifolds and the Hermitian-Einstein equation, we are now ready to present our goal of this chapter. Let us begin by showing the definition of compact *Kähler* manifold, denoted $(\mathcal{X},\omega_\mathcal{X})$.

Definition 6.3.1. A *compact Kähler manifold* is a n-dimensional complex manifold $\mathcal{X}$ that is compact as a topological space and equipped with a fixed holomorphic tangent bundle, $T^{1,0}\mathcal{X}$ associated with the *Kähler* form (i.e. (1,1)-form), such that, $\omega = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$ is closed. Note that the compactness simply means that the every open cover of $\mathcal{X}$ has a finite subcover. Such examples of *compact Kähler manifolds* are complex projective space: $\mathbb{CP}^n$ and Calabi-Yau manifolds.

Definition 6.3.2. Let $\mathcal{X}$ be a compact *Kähler* manifold. A holomorphic vector bundle over a compact *Kähler* manifold, $\mathcal{E} \to \mathcal{X}$, is a vector bundle associated with a fixed Hermitian metric $H_{\bar{\alpha}\beta}$ and a base metric $\omega = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$,which is *Kähler.*

Recall that from chapter 4, we have seen that the *Yang-Mills equation* is $\nabla^p F_{pq}{}_{\beta}^{\alpha} = 0$, for a unitary connection $\mathcal{A}$ with its curvature $\mathcal{F}$. Then we first claim that such equation is integrable, since our base manifold is a *Kähler* manifold.

Definition 6.3.3. A connection $\mathcal{A}$ is integrable if its curvature is a (1,1)-form valued in $End(\mathcal{E})$.

Now, from chapter 3 on characteristic classes, any curvature satisfies the Second Bianchi Identity. To to this, we will introduce useful basic lemma and theorem.

Lemma 6.3.4. *Suppose that the curvature $\mathcal{F}$ is a (1,1)-form. Then the connection $\mathcal{A}$ equipps with a vector bundle, $\mathcal{E} \to \mathcal{X}$, having the holomorphic structure, where holomorphic sections are defined by $\nabla_{\bar{k}}\varphi^{\alpha} = 0$ with $1 \leq k \leq rank(\mathcal{E})$.*

Proof. On a complex manifold $\mathcal{X}$, it is clear that each 1-from, Λ^1, decomposes into (1,0) and (0,1) forms. Then we can write our connection $\mathcal{A} = \mathcal{A}^{1,0} + \mathcal{A}^{0,1}$. Explicitly, by the definition of Chern-Unitary connection:

$$\nabla_k \varphi^{\alpha} = H^{\alpha\bar{\beta}} \partial_k (H_{\bar{\beta}\gamma}\varphi^{\gamma}) \text{ and } \nabla_{\bar{k}}\varphi^{\alpha} \in \Gamma(\mathcal{X}, \mathcal{E} \otimes \Lambda^{0,1})$$

$\nabla_{\bar{k}}$ is a connection of (0,1)-form. Also, we alredy know that for the curvature of 2-form valued in $End(\mathcal{E})$ is $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \mathcal{F}^{2,0} + \mathcal{F}^{1,1} + \mathcal{F}^{0,2}$. Then, by assumption, $\mathcal{F}$ is 1-1 form, which implies $\mathcal{F}^{0,2} = 0$. This follows that $(\nabla_{\bar{k}})^2 = 0$. Specifically, such holomorphic sections of $\mathcal{E}$ can be characterized by $\nabla_{\bar{k}}\varphi^{\alpha} = 0$ for all $\bar{k}$ and α. Hence, by the definition 6.3.3. and our observation, we have shown that the connection $\nabla_{\bar{k}}$ of (0,1)-form is said to be integrable is $(\nabla_{\bar{k}})^2 = 0$. Q.E.D.

We can expand this lemma to the following theorem.

Theorem 6.3.5. *If the curvature $\mathcal{F}$ is a (1,1)-form, then there exists enough holomorphic sections in order to trivialize the vector bundle, and hence indeed the bundle itself is a holomorphic vector bundle.*

Proof. The proof immediately follows from the definition 2.1.3. and the lemma 6.3.4. Since we have the curvature $\mathcal{F}$ of (1,1)-form, we can consturct local holomorphic frams $\{\varphi^1, \ldots, \varphi^r\}$ for $rank(\mathcal{E})$, in which satisfies the integrability $\nabla_{\bar{k}}\varphi^{\alpha} = 0$. Q.E.D.

Now, we will apply the lemma 6.3.4. and the theorem 6.3.5. to demonstrate the Second Bianchi Identity on *Kähler* manifolds.

Theorem 6.3.6. *Let $\mathcal{X}$ be a compact Kähler manifold. Suppose that $\mathcal{E} \to \mathcal{X}$ be a smooth complex vector bundle, $\mathcal{A}$ be the connection on $\mathcal{E}$, and $\mathcal{F}$ be its curvature. Then the Second Bianchi Identity holds:*

$$d_{\mathcal{A}}\mathcal{F} = 0$$

Proof. We have two versions. The first one is very directforward, followed by definition 3.1.1. We expand the identity $d_{\mathcal{A}}\mathcal{F} = 0$ explicitly as $\nabla\mathcal{F} = 0$, where ∇ is the covariant derivative on $End(\mathcal{E})$

$$d_{\mathcal{A}}\left(\frac{1}{2}\sum_{j,k}\mathcal{F}_{kj}dx^j \wedge dx^k\right) = \frac{1}{2}\sum_{j,k,l}\nabla_l \mathcal{F}_{kj} dx^l \wedge dx^j \wedge dx^k = 0$$

However, this is not convenient to use as we have just applied the expansion. Since $\mathcal{X}$ is a *Kähler* manifold, we can see the curvature $\mathcal{F}$ is indeed a (1,1)-form. So, by the lemma 6.3.4. above, in holomorphic coordinates z^j for $\mathcal{X}$, where $\mathcal{F} = \sum_{j,k}\mathcal{F}_{\bar{k}j} dz^j \wedge d\bar{z}^k$, such equation simplifies to

$$d_{\mathcal{A}}(\mathcal{F}_{\bar{k}j}dz^j \wedge d\bar{z}^k) = \nabla_m \mathcal{F}_{\bar{k}j} dz^m \wedge dz^j \wedge d\bar{z}^k + \nabla_{\bar{m}}\mathcal{F}_{\bar{k}j} d\bar{z}^m \wedge dz^j \wedge d\bar{z}^k = 0$$

This means, we can differentiate m and $\bar{m}$ directions for the covariant derivatives. In fact, since our situation is in complex case, each form is not commutative. We observe that the first part: $dz^m \wedge dz^j \wedge d\bar{z}^k$ is a (2,1)-form and the second part: $d\bar{z}^m \wedge dz^j \wedge d\bar{z}^k$ is a (1,2)-form. Then we find each part must be zero, which is now

$$\sum \nabla_m \mathcal{F}_{\bar{k}j} dz^m \wedge dz^j \wedge d\bar{z}^k = 0 \text{ and } \sum \nabla_{\bar{m}} \mathcal{F}_{\bar{k}j} d\bar{z}^m \wedge dz^j \wedge d\bar{z}^k = 0$$

By the antisymmetrization, we have shown the Second Bianchi Identity in *Kähler* manifolds

$$\nabla_m \mathcal{F}_{\bar{k}j} - \nabla_j \mathcal{F}_{\bar{k}m} = 0 \text{ and } \nabla_{\bar{m}} \mathcal{F}_{\bar{k}j} - \nabla_{\bar{k}} \mathcal{F}_{\bar{m}j} = 0$$

where ∇ is the connection valued on $End(\mathcal{E})$. Q.E.D.

Next, we show another version of the proof, which is more advanced. We use the result of the theorem 6.3.5 in which the holomorphic trivialization ensures the Second Bianchi Identity.

Proof. Let us denote $\mathcal{D}$ be the new connection, acting on $\Lambda^{1,0} \otimes End(\mathcal{E})$. i.e. We take the connection ∇ on $End(\mathcal{E})$, together with the Chern-unitary connection on $\Lambda^{1,0}$, given by $\Gamma^p_{mq} = g^{p\bar{r}} \partial_m g_{\bar{r}q}$. Then, we show the relations including indices of the curvature $\mathcal{F}$ and $End(\mathcal{E})$:

$$\mathcal{D}_m \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} = \nabla_m \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} - \mathcal{F}_{\bar{k}p}{}^{\alpha}{}_{\beta} \Gamma^p_{mj} \text{ and } \mathcal{D}_{\bar{m}} \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} = \nabla_{\bar{m}} \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} - \mathcal{F}_{\bar{k}q}{}^{\alpha}{}_{\beta} \Gamma^q_{\bar{m}j}$$

Now, we can express the Second Bianchi Identity in terms of D in both m and $\bar{m}$ directions:

$$\mathcal{D}_m \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} - \mathcal{D}_j \mathcal{F}_{\bar{k}m}{}^{\alpha}{}_{\beta} - \mathcal{F}_{\bar{k}p}{}^{\alpha}{}_{\beta} \Gamma^p_{mj} + \mathcal{F}_{\bar{k}p}{}^{\alpha}{}_{\beta} \Gamma^p_{jm} = 0 \text{ by the antisymmetry of } \mathcal{F} \text{ (swap } m \leftrightarrow j)$$

and similarly,

$$\mathcal{D}_{\bar{m}} \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} - \mathcal{D}_{\bar{k}} \mathcal{F}_{\bar{m}j}{}^{\alpha}{}_{\beta} - \mathcal{F}_{\bar{k}q}{}^{\alpha}{}_{\beta} \Gamma^q_{\bar{m}j} + \mathcal{F}_{\bar{k}q}{}^{\alpha}{}_{\beta} \Gamma^q_{j\bar{m}} = 0 \text{ again, by the antisymmetry, } \left(\text{swap } \bar{m} \leftrightarrow \bar{k}\right)$$

Notice that the last two terms of the equations are $-\mathcal{F}_{\bar{k}p}{}^{\alpha}{}_{\beta}\left(\Gamma^p_{mj} - \Gamma^p_{jm}\right)$ and $-\mathcal{F}_{\bar{k}q}{}^{\alpha}{}_{\beta}\left(\Gamma^q_{\bar{m}j} - \Gamma^q_{j\bar{m}}\right)$. Since our base metric $g_{\bar{k}j}$ is *Kähler*, the connection is clearly torision free. Which implies $\Gamma^p_{mj} = \Gamma^p_{jm}$ and $\Gamma^q_{\bar{m}j} = \Gamma^q_{j\bar{m}}$; hence, each term vanishes. Thus, the Second Bianchi Identity becomes:

$$\mathcal{D}_m \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} = \mathcal{D}_j \mathcal{F}_{\bar{k}m}{}^{\alpha}{}_{\beta} \text{ and } \mathcal{D}_{\bar{m}} \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} = \mathcal{D}_{\bar{k}} \mathcal{F}_{\bar{m}j}{}^{\alpha}{}_{\beta}$$

Q.E.D.

By understanding the Second Bianchi Identity on compact *Kähler* manifolds, we are now ready to derive the integration of the *Yang-Mills equation.*

Theorem 6.3.7. *The Yang-Mills equation on holomorphic vector bundles over compact Kähler maniolfds reduces to Hermitian-Einstein equation.*

Proof. Recall that we have the *Yang-Mills* equation

$$\nabla^p \mathcal{F}_{pq}{}^{\alpha}{}_{\beta} = 0 \tag{6.3}$$

Suppose that $\mathcal{E} \to \mathcal{X}$ is holomorphic vector bundles and $(\mathcal{X}, \omega_{\mathcal{X}})$ is *Kähler* with z^j be holomorphic coordinates for the basis of $\mathcal{X}$. Then we can permute indices p and j followed by the result of the proposition 6.1.4. So, the equation splits into

$$\nabla^{\bar{k}} \mathcal{F}_{\bar{k}j}{}^{\alpha}{}_{\beta} = 0 \text{ and } \nabla^k \mathcal{F}_{\bar{j}k}{}^{\alpha}{}_{\beta} = 0$$

This is because, our curvature $\mathcal{F}$ is a (1,1)-form, whose only non-zero components are $\mathcal{F}_{\bar{k}j\,\beta}^{\;\;\alpha}$. i.e. we can contract k and $\bar{k}$. Which implies, for $\nabla_{\bar{k}}$, we can differentiate with respect to p and raise the index $g^{p\bar{k}}$. We find

$$\nabla^{\bar{k}}\mathcal{F}_{\bar{k}j\,\beta}^{\;\;\alpha} = g^{p\bar{k}}\nabla_p\mathcal{F}_{\bar{k}j\,\beta}^{\;\;\alpha} = 0$$

,and by the Second Bianchi Identity, we permute p and j

$$g^{p\bar{k}}\nabla_p\mathcal{F}_{\bar{k}j\,\beta}^{\;\;\alpha} = g^{p\bar{k}}\nabla_j\mathcal{F}_{\bar{k}p\,\beta}^{\;\;\alpha}$$

Since ∇_p is the covariant derivative to all the variables, we put the metric inside and get

$$\nabla^{\bar{k}}\mathcal{F}_{\bar{k}j\,\beta}^{\;\;\alpha} = g^{p\bar{k}}\nabla_j\mathcal{F}_{\bar{k}p\,\beta}^{\;\;\alpha} = \nabla_j\left(g^{p\bar{k}}\mathcal{F}_{\bar{j}p\,\beta}^{\;\;\alpha}\right) = 0$$

Here, notice that $g^{p\bar{k}}\mathcal{F}_{\bar{j}p\,\beta}^{\;\;\alpha} = \Lambda\mathcal{F}$, which is the definiton of the Hermitian-Einstein equation. Hence, we now have

$$\nabla^{\bar{k}}\mathcal{F}_{\bar{k}j\,\beta}^{\;\;\alpha} = \nabla_j(\Lambda\mathcal{F}) = 0$$

Similarly, by the same analogue, we permute anti-holomorphic indices $\bar{p}$ and $\bar{j}$, and thus conclude to $0 = \nabla_{\bar{p}}(\Lambda\mathcal{F})$. Finally, by taking the *Yang-Mills* equation, the solutions are clearly integrable in the complex structure. So, the covariant derivative of any direction $\bar{p}$ or j of $\Lambda\mathcal{F}$ is zero, which is equivalent to say $\Lambda\mathcal{F}$ is a constant. Thus, we have shown that the *Yang-Mills* equation deduces into the Hermitian-Einstein equation $\Lambda\mathcal{F} = \mu I$.

$$\nabla^p\mathcal{F}_{pq\,\beta}^{\;\;\alpha} = 0 \Rightarrow \nabla(\Lambda\mathcal{F}) = 0 \Rightarrow \Lambda\mathcal{F} \text{ is constant } \Rightarrow \Lambda\mathcal{F} \text{ is a scalar multiple of the identity}$$

Q.E.D.

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